REPRESENTATIONS OF THE EXCEPTIONAL LIE SUPERALGEBRA $E(3,6)$ III: CLASSIFICATION OF SINGULAR VECTORS.

VICTOR G. KAC * AND ALEXEI RUDAKOV

Abstract. We continue the study of irreducible representations of the exceptional Lie superalgebra $E(3,6)$. This is one of the two simple infinite-dimensional Lie superalgebras of vector fields which have a Lie algebra $s\ell(3) \times s\ell(2) \times g\ell(1)$ as the zero degree component of its consistent $\mathbb{Z}$-grading. We provide the classification of the singular vectors in the degenerate Verma modules over $E(3,6)$, completing thereby the classification and construction of all irreducible $E(3,6)$-modules that are L_0-locally finite.

0. Introduction.

There are only two infinite-dimensional simple linearly compact Lie superalgebras, $E(3,6)$ and $E(3,8)$, that have the Lie algebra $s\ell(3)\times s\ell(2)\times g\ell(1)$ as the zero degree component $\mathfrak{g}_0$ in their consistent $\mathbb{Z}$-grading [2]. In the present paper we continue the study of irreducible representations of $E(3,6)$, the Lie superalgebra which has apparent relations to the Standard Model (see [4]).

The article is a sequel to our papers [3, 4] and together they provide the classification and description of all irreducible L_0-locally finite representations of $E(3,6)$ (see [3] for the definition).

As was shown in [3] the problem can be solved in two steps. First one obtains the classification of the so called degenerate (generalized) Verma modules (the Verma modules that have non-trivial singular vectors), and then one describes the irreducible factors of these degenerate Verma modules.

In [4], both probelms were solved modulo the classification of singular vectors. In this article we provide the proof of this classification (the result was announced in [4]). Thus we complete the classification of the irreducible $E(3,6)$-modules that are L_0-locally finite.

As before all vector spaces, linear maps and tensor products are considered over the field $\mathbb{C}$ of complex numbers.

1. Notations and basic basic properties of $E(3,6)$.

We recall here the basic definitions and notations from [1,2,3].

* Supported in part by NSF grant DMS-0201017.

To construct $E(3,6)$ we use its embedding into $E(5,10)$. Namely consider even variables $x_1,\dots,x_5$. We also use the notations $z_+ = x_4$, $z_- = x_5$ further on. Let S_5 be the Lie algebra of divergence zero formal vector fields in these variables, and $\mathsf{d}\Omega^1(5)$ the space of closed (=exact) formal differential 2-forms in these variables.

Let us remind that the Lie superalgebra $E(5,10)$ has $E(5,10)_{\bar 0} \simeq S_5$ as a Lie algebra, $E(5,10)_{\bar 1} \simeq \mathsf{d}\Omega^1(5)$ as an S_5-module and the brackets on $E(5,10)_{\bar 1}$ are defined via the exterior product of differential forms.

We shall be using the following notations:

$$\mathsf{d}_{jk} := \mathsf{d}x_j \wedge \mathsf{d}x_k, \quad \mathsf{d}_i^+ := \mathsf{d}_{i4} \quad \mathsf{d}_i^- := \mathsf{d}_{i5} \qquad \partial_i := \partial/\partial x_i \quad \partial_+ := \partial_4, \quad \partial_- := \partial_5.$$

An element A from $E(5,10)_{\bar 0} = S_5$ can be written as

$$A = \sum_i a_i \partial_i\,, \qquad \text{where } a_i \in \mathbb{C}\,[[x_1,\dots,x_5]]\,, \quad \sum_i \partial_i a_i = 0\,,$$

and an element B from $E(5,10)_{\bar 1}$ is of the form

$$B = \sum_{j,k} b_{jk}\mathsf{d}_{jk}\,, \text{ where } b_{jk} \in \mathbb{C}[[x_1,\dots,x_5]],\, dB = 0\,.$$

The brackets in $E(5,10)_{\bar 1}$ can be computed using bilinearity and the rule

$$[a\mathsf{d}_{jk}, b\mathsf{d}_{lm}] = \varepsilon_{ijklm} ab\partial_i$$

where ε_{ijklm} is the sign of the permutation $(ijklm)$ when $\{i,j,k,l,m\}$ are distinct and zero otherwise.

We have for $L = E(3,6)$ the following description of the first three pieces of its consistent $\mathbb{Z}$-grading $L = \Pi_{j\ge -2}\,\mathfrak{g}_j$:

$$\mathfrak{g}_{-2} = \langle \partial_i,\, i = 1,2,3\rangle, \quad \mathfrak{g}_{-1} = \langle \mathsf{d}_{ij},\, i = 1,2,3,\, j = 4,5\rangle.$$

And $\mathfrak{g}_0 = s\ell(3) \oplus s\ell(2) \oplus g\ell(1)$ with the following basis:

$$\begin{aligned}
&h_1 = x_1\partial_1 - x_2\partial_2, \quad h_2 = x_2\partial_2 - x_3\partial_3, \quad e_1 = x_1\partial_2, \quad e_{12} = x_2\partial_3, \quad e_3 = x_1\partial_3,\\
&f_1 = x_2\partial_1,\ \ f_2 = x_3\partial_2,\ \ f_{12} = x_3\partial_1,\ \ h_3 = x_4\partial_4 - x_5\partial_5,\ \ e_3 = x_4\partial_5, \quad f_3 = x_5\partial_4,\\
&Y = \tfrac{2}{3}(x_1\partial_1 + x_2\partial_2 + x_3\partial_3) - (x_4\partial_4 + x_5\partial_5).
\end{aligned}$$

We keep the standard Cartan subalgebra $\mathcal{H} = \langle h_1, h_2, h_3, Y\rangle$ and the standard Borel subalgebra $\mathcal{B} = \mathcal{H} \oplus \mathcal{N}$, where $\mathcal{N} = \langle e_1, e_2, e_{12},\, e_3\rangle$, of $\mathfrak{g}_0$. We denote by $\mathrm{wt}_2\, v$ the $s\ell(2)$-weight of v and by $\mathrm{wt}_3\, v$ the $s\ell(3)$-weight whenever the weight is defined. We denote by $\mathrm{wt}_1\, v$ the eigenvalue of Y on v whenever defined.

The algebra $E(3,6)$ is generated by $\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1$ moreover it is generated by $\mathfrak{g}_0$ and the three elements e_0, f_0 and e_0^- (the notation of the last element in [3] was e_0') from the following four:

$$\begin{aligned}
f_0 &= \mathsf{d}_{14},\\
e_0^- &= x_3\mathsf{d}_{35},\\
e_0^+ &= x_3\mathsf{d}_{34},\\
e_0 &= x_3\mathsf{d}_{25} - x_2\mathsf{d}_{35} + 2x_5\mathsf{d}_{23}\,.
\end{aligned}$$

It is known ([3]) that the element f_0 is the highest weight vector of the g_0-module g_{-1}, while e_0^-, e_0 are the lowest weight vectors of the g_0-module g_1 and

$$[e_0^-, f_0] = f_2, \tag{1.1}$$

$$[e_0, f_0] = \tfrac{2}{3}h_1 + \tfrac{1}{3}h_2 - h_3 - Y =: h_0. \tag{1.2}$$

We use the notations $\mathsf{g}_{-1}^{\pm} = \langle \mathsf{d}_1^{\pm}, \mathsf{d}_2^{\pm}, \mathsf{d}_3^{\pm} \rangle$, and the following ones for the elements from the exterior algebras $\Lambda^{\pm} := \Lambda \mathsf{g}_{-1}^{\pm}$:

$$\mathsf{d}_{ij}^{\pm} := \mathsf{d}_i^{\pm} \cdot \mathsf{d}_j^{\pm}, \qquad \mathsf{d}_{ijk}^{\pm} := \mathsf{d}_i^{\pm} \cdot \mathsf{d}_j^{\pm} \cdot \mathsf{d}_k^{\pm}.$$

It is easy to see ([3]) that the following subalgebras $\mathsf{g}_1^{\pm}$ are abelian and are normalized by $s\ell(3)$:

$$\mathsf{g}_1^{\pm} = \langle x_i \mathsf{d}_j^{\pm} + x_j \mathsf{d}_i^{\pm} \,|\, i,j = 1,2,3 \rangle. \tag{1.3}$$

Note that $[\mathsf{g}_{-1}^{\pm}, \mathsf{g}_1^{\mp}] = 0$, and that the subalgebras

$$\mathcal{S}^{\pm} = \mathsf{g}_{-1}^{\pm} \oplus s\ell(3) \oplus \mathsf{g}_1^{\pm}.$$

are each isomorphic to the simple Lie superalgebra $S(0|3)$ of divergenceless vector fields in three anticommuting indeterminates.

It is important for us that the change of the variables $x_i \to x_i,\ i \leq 3,\ x_4 \to x_5$, $x_5 \to x_4$ induces an automorphism φ of the algebra $L = E(3,6)$. Clearly φ preserves the grading and the subalgebras $s\ell(3)$, $s\ell(2)$, $g\ell(1)$ of g_0. The restriction of φ on $s\ell(3) \oplus g\ell(1)$ is the identity map, but on $s\ell(2)$ we have

$$\varphi\, e_3 = f_3, \quad \varphi\, f_3 = e_3, \quad \varphi\, h_3 = -h_3. \tag{1.4}$$

Also φ interchanges $\mathcal{S}^{\pm}$ and its action on L_- is defined by the formulae

$$\varphi\, \mathsf{d}_i^{\pm} = \mathsf{d}_i^{\mp}, \quad \varphi\, \widehat{\partial}_i = -\widehat{\partial}_i \quad \text{for } i = 1,2,3. \tag{1.5}$$

2. **Contraction and quasi-singular vectors.**

We use notation $L_- = \oplus_{j<0}\, \mathsf{g}_j$, $\quad L_+ = \Pi_{j>0}\, \mathsf{g}_j$, $\quad L_0 = \mathsf{g}_0 \oplus L_+$. Of course

$$L = L_- \oplus \mathsf{g}_0 \oplus L_+,$$

Our main objective is to study singular vectors in the generalized Verma modules

$$M(\mathbf{V}) = U(L) \otimes_{U(L_0)} \mathbf{V} \cong U(L_-) \otimes_{U(\mathsf{g}_0)} \mathbf{V}, \tag{2.1}$$

where as usual $\mathbf{V}$ is a finite-dimensional irreducible g_0-module extended to L_0 by letting g_j for $j > 0$ acting trivially.

We are concerned with singular vectors in $M(\mathbf{V})$ that are also the highest weight vectors with respect to the standard Cartan and Borel subalgebras $\mathcal{H}$ and $\mathcal{B}$ of g_0. By Proposition 2.2 of [3] the defining property of these vectors can be written as follows:

(2.2) *The g_0-highest weight vector $\mathbf{v}$ of a $E(3,6)$-module is singular iff*
$$e_0 \cdot \mathbf{v} = 0, \qquad e_0^- \cdot \mathbf{v} = 0.$$

Property $e_0^- \cdot \mathbf{v} = 0$ implies $e_0^+ \cdot \mathbf{v} = 0$ when $\mathbf{v}$ is an $s\ell(2)$-highest weight vector, but in general these are different conditions. For $s\ell(3)$-highest weight vectors the first amounts to being $\mathcal{S}^-$-singular (i.e. killed by $\mathfrak{g}_1^-$) and the second to being $\mathcal{S}^+$-singular (i.e. killed by $\mathfrak{g}_1^+$).

Definition 2.1. *We call a vector v quasi-singular vector if it is $s\ell(3)$-highest weight vector and $\mathcal{S}^-$-singular. We call a vector v semi-singular vector if it is $s\ell(3)$-highest weight vector and is both $\mathcal{S}^-$ and $\mathcal{S}^+$-singular.*

In the sequel we shall often say *highest vector* in place of *highest weight vector*. Of course an $s\ell(3)$-highest singular vector is both semi-singular and quasi-singular. Also the $\mathfrak{g}_0$-highest vector of an $E(3,6)$-module is singular if only only if it is quasi-singular and is annihilated by e_0. But for a vector $\mathbf{v}$ that is not $\mathfrak{g}_0$-highest the property $e_0^- \cdot \mathbf{v} = 0$ does not in general imply $e_0^+ \cdot \mathbf{v} = 0$, the condition of being quasi-singular is indeed weaker.

We will first describe quasi-singular vectors, then narrow the “list of suspects” to semi-singular ones, and then come to the description of the $\mathfrak{g}_0$-highest singular vectors.

It is important to mention that we can consider quasi-singular vectors in a space that is not necessary $E(3,6)$-module, but only $\mathcal{S}^-$-module.

For a vector space T and a linear form $\tau : T \longrightarrow \mathbb{C}$, $\tau(t) = \langle t|\tau\rangle$, we shall use the contraction maps defined as follows. For any linear space A the contraction map $\langle\tau\rangle : A \otimes T \longrightarrow A$ applied to $a \otimes t$ gives $\langle t|\tau\rangle a$.

Remark 2.2. *If both A and T are $s\ell(2)$-modules, $w \in A \otimes T$ is a weight vector of weight λ and a form τ has weight μ, then the contraction $w\langle\tau\rangle$ has weight $\lambda + \mu$. When w and τ are eigenvectors for Y, the contraction $v = w\langle\tau\rangle$ is also an eigenvector (with eigenvalue equals to the sum of the eigenvalues).*

Let $\mathbf{V}$ be a (finite-dimesional) $\mathfrak{g}_0$-module that is isomorphic to the tensor product of $s\ell(3) \oplus g\ell(1)$-module V and $s\ell(2)$-module T (i.e.$\mathbf{V} = V \otimes T$). Denote by $F(p,q)$ the irreducible representation of $s\ell(3)$ with the highest weight (p,q).

Remark 2.3. *The main theorem from [3] states that the non-trivial highest singular vector $\mathbf{v} \in M(\mathbf{V})$ could exist only when the $s\ell(3)$-module V is a submodule of $\mathbb{C}[x_1, x_2, x_3]$ or $\mathbb{C}[\partial_1, \partial_2, \partial_3]$. Therefore we shall look for quasi-singular and semi-singular vectors only in $U(L_-) \otimes_{\mathbb{C}} \mathbf{V}$, where $\mathbf{V} = V \otimes T$, and V belongs to $\mathbb{C}[x_1, x_2, x_3]$ or $\mathbb{C}[\partial_1, \partial_2, \partial_3]$, or for that matter we can suppose that either $V = F(p,0)$ or $V = F(0,q)$.*

Let us notice that there is a natural $g\ell(1)$-action on $\mathbb{C}[x_1, x_2, x_3]$ and $\mathbb{C}[\partial_1, \partial_2, \partial_3]$ such that

$$Y\, x_i = \tfrac{2}{3} x_i\,, \quad Y\, \partial_i = -\tfrac{2}{3} \partial_i\,. \tag{2.3}$$

Remark 2.4. *We will assume that $s\ell(2)$ acts trivially on V and $g\ell(1)$ acts according to the above rule, thus that wt_2, wt_1 are defined on the tensor product $M(V) = U(L_-) \otimes V$.*

Let us notice that the tensor product of $s\ell(3)$-modules $U(L_-)\otimes V$ has natural $\mathcal{S}^{\pm}$-structures and that there is a natural isomorphism

$$M(\mathbf{V}) \cong (U(L_-)\otimes V)\otimes T \tag{2.4}$$

so *the contraction map* for any $\tau \in T^*$ is well defined. We shall use this map further on. It is of course the homomorphism of $\mathcal{S}^-$ (or $\mathcal{S}^+$)-modules. The following result will play a key role in our subsequent calculation of singular vectors.

Proposition 2.5. *Given a non-zero* $\mathfrak{g}_0$*-highest singular vector* $\mathbf{v}\in M(V\otimes T)$ *consider a contraction* $v=\mathbf{v}\langle\tau\rangle \in U(L_-)\otimes V$*, where* $\tau\in T^*$*.*

1. *For any* τ*, the contraction* v *is a semi-singular vector.*
2. *Whenever* τ *is an* $s\ell(2)$*-weight vector,* v *is such a vector too.*
3. *There exists* $\tau\in T^*$ *such that* v *is a non-zero semi-singular vector of non-negative* $s\ell(2)$*-weight, and* v *is also a* $g\ell(1)$*-weight vector.*

Proof. Whatever τ, the contraction clearly commutes with the $s\ell(3)$-action, therefore we get the $s\ell(3)$-highest vector from $s\ell(3)$-highest one.

Because

$$[e_0^{\pm},\, \mathfrak{g}_{-1}] \subset s\ell(3) \subset \mathfrak{g}_0 \tag{2.5}$$

the $\mathcal{S}^+$ and $\mathcal{S}^-$-structures are defined on $U(L_-)\otimes V$. The operators $e_0^{\pm}$ act on both sides of the contraction map, and commute with the map . Thus v will be annihilated by $e_0^{\pm}$ because $\mathbf{v}$ is, hence v is semi-singular.

To finish the proof, notice that for any non-zero $s\ell(2)$-highest vector in $A\otimes T$ there always exists a contraction such that the result is a non-zero vector of non-negative weight. This follows immediately from the formula for the highest weight vectors in the tensor product of irreducible (finite-dimensional) $s\ell(2)$-modules. □

Definition 2.6. *We say that a semi-singular (resp. quasi-singular) vector is admissible if the vector has a non-negative* $s\ell(2)$*-weight.*

Proposition 2.5 and Remark 2.3 show that we should look for admissible quasi-singular and semi-singular vectors in $U(L_-)\otimes_{\mathbb{C}} V$ where V belongs to $\mathbb{C}[x_1,x_2,x_3]$ or $\mathbb{C}[\partial_1,\partial_2,\partial_3]$. We provide the description of these vectors in the next section after introducing appropriate notations.

Remark 2.7. *Let us notice that the authomorphism* φ *naturally extends to* $U(L_-)\otimes_{\mathbb{C}} V$*, where it acts on* $U(L_-)$ *according to formulae* (1.5) *and on* V *identically. Evidently a vector* $\varphi(w)\in U(L_-)\otimes_{\mathbb{C}} V$ *is semi-singular, if and only if* w *is semi-singular. Because of* (1.4) *we get* $\mathrm{wt}_2\, w = -\mathrm{wt}_2\,\varphi(w)$*. Therefore applying* φ *we shall get all semi-singular vectors as soon as we know the admissible ones.*

3. The description of admissible quasi-singular and semi-singular vectors.

We need elaborated notations and some lemmas before starting the explicit calculation of the quasi-singular and semi-singular vectors. We keep the following notations:

$$\mathtt{S}^k := \mathrm{Sym}^k(\mathfrak{g}_{-2}), \quad \mathtt{S} = \sum_{k\geq 0} \mathtt{S}^k, \quad \Lambda_i^{\pm} := \Lambda^i(\mathfrak{g}_{-1}^{\pm}).$$

As in [3] we use "a hat" to mark the elements of $\mathfrak{g}_{-2} \subset \mathtt{S} \subset U(L)$, thus further on

$$\mathtt{S} = \mathbb{C}[\,\widehat{\partial}_1, \widehat{\partial}_2, \widehat{\partial}_3].$$

We will follow the approach developed in [3] for working with the highest vectors in tensor products of $sl(3)$-modules where the first factor is $\mathtt{S}$.

Let us refine the isomorphism in (2.1) to an $s\ell(3)$-isomorphism of $M(V)$ with the tensor products of $s\ell(3)$-modules (we omit the tensor product signs):

$$M(V) \cong \mathtt{S}\,\Lambda^-\Lambda^+\,V. \tag{3.1}$$

We use this isomorphism later on. Also for later use we introduce the following notations.

We denote $\deg_\Lambda v$ the combined degree of $\Lambda^-\Lambda^+$-part in $w \in \mathtt{S}\,\Lambda^-\Lambda^+\,V$, for example $\deg_\Lambda \widehat{\partial}_1\widehat{\partial}_2\,\mathsf{d}_1^-\mathsf{d}_{12}^+\,v = 3$. Let us mention that $\mathrm{wt}_2\,w$ depends only on the $\Lambda^-\Lambda^+$-part of w, for example $\mathrm{wt}_2\,\mathsf{d}_1^-\mathsf{d}_{12}^+\,v = \mathrm{wt}_2\,\widehat{\partial}_1\widehat{\partial}_2\,\mathsf{d}_1^-\mathsf{d}_{12}^+\,v = +1$.

For any space M given with the isomorphism $M \cong \mathtt{S} \otimes W$, whatever W, we can think of an element $f \in M$ as a polynomial in $\widehat{\partial}_i$ with coefficients from W written to the right of these "variables". We define $\deg_{\mathtt{S}} f$ to be the *degree* of such a polynomial f. We also define subspaces $\mathbf{lex}(< a)M$ (resp. $\mathbf{lex}(\leq a)M$), for a multi-index a, to be the linear span of monomials $\widehat{\partial}^b \cdot w$, any $w \in W$ and $b <_{\mathrm{lex}} a$, (resp. $b \leq_{\mathrm{lex}} a$). As usual the expression

$$f \equiv g \mod \mathbf{lex}(< a)$$

means that $f - g \in \mathbf{lex}(< a)M$, and we omit M when it is clear from the context. Similarly

$$f \equiv g \mod \deg_{\mathtt{S}}(< N)$$

means that $\deg_{\mathtt{S}}(f - g) < N$.

As in [3] we say that $\widehat{\partial}^a \cdot w \neq 0$ is the *lexicographically highest term* of f iff

$$f \equiv \widehat{\partial}^a \cdot w \mod \mathbf{lex}(< a).$$

Similar notations will be used for the *degree-lexicographic* order on the monomials that is defined by the condition

$$\partial^b \leq_{\mathrm{dlex}} \partial^a \iff (|b|, b) \leq_{\mathrm{lex}} (|a|, a), \quad \text{where } |a| = \deg_{\mathtt{S}} \partial^a = \sum a_i. \tag{3.2}$$

Let s be an integer, denote $h'\{s\} = h_1 + h_2 + s + 1$, $h_2\{s\} = h_2 + s$. Slightly generalizing the definition in Section 3 of [3] let B, $D_i\{s\} \in \mathtt{S}\#U(sl(3))$ be:

$$
\begin{aligned}
B \ &= f_{12}h_1 + f_2 f_1 = B = f_{12}(h_1+1) + f_1 f_2,\\
D_1\{s\} &= \widehat{\partial}_1\, h_1 h'\{s\} + \widehat{\partial}_2\, f_1 h'\{s\} + \widehat{\partial}_3\, B,\\
D_2\{s\} &= \widehat{\partial}_2\, h_2\{s\} + \widehat{\partial}_3\, f_2,\\
D_3\{s\} &= \widehat{\partial}_3.
\end{aligned}
\tag{3.3}
$$

We write simply D_i when $s = 0$. We also use the usual multi-index notations

$$D^a\{s\} := D\{s\}^a = D_1\{s\}^{a_1} D_2\{s\}^{a_2} D_3\{s\}^{a_3}.$$

Let us repeat the basic facts about these operators from [3].

Lemma 3.1. *The operators $D_i\{s\}$ (with the same s) commute with each other.*

Proof. The proof is identical to that of Lemma 3.7 in [3], using that s does not change the commutators. □

Proposition 3.2. *Let M be a finite-dimensional irreducible $s\ell(3)$-module with the highest vector m_0 of weight $\mathrm{wt}_3\, m_0 = \mu$. Any monomial $D^a m_0$ provides us with the highest vector in the tensor product of $s\ell(3)$-modules $\mathtt{S} \otimes M$ that has the $s\ell(3)$-weight given by the formula*

$$\mathrm{wt}_3\, D^\alpha\{s\} m_0 = \mu + \sum a_i\, \mathrm{wt}_3\, \partial_i = \mu + a_1(-1,0) + a_2(1,-1) + a_3(0,1),$$

(whenever the expression for $\mathrm{wt}_3\, D^\alpha m_0$ gives non-dominant weight, one has $D^\alpha m_0 = 0$). Any highest weight vector in $\mathtt{S} \otimes M$ can be written uniquely as the following linear combination

$$w = \sum_\alpha c_\alpha D^\alpha m_0\,, \quad c_\alpha \in \mathbb{C}.$$

We have combined here the results proven at several places in Section 3 of [3].

In the proposition below we provide an explicit formula for $D^\alpha\{s\}$, but we state two lemmas first.

In the following $A^{[n]} := A(A-1)\cdots(A-n+1)$.

Lemma 3.3. $(x-r)^{[m]} = \sum_{i=0}^m (-1)^i \binom{m}{i} r^{[i]} (x-i)^{[m-i]}$.

Proof. Induction on m using the fact that

$$(x-n)(x-r)^{[n]} - n((x-1)-(r-1))^{[n]} = (x-r)^{[n+1]}.$$

□

Lemma 3.4. *Let $D = ah + \partial b$ where*

$$
\begin{aligned}
&[a,b] = 0, && [\partial, b] = +a, && [\partial, a] = 0\,,\\
&[h,b] = -2b, && [h,a] = -a, && [h,\partial] = \partial\,.
\end{aligned}
$$

Then

$$D^k = \sum_{m=0}^k \binom{k}{m} a^{k-m} \partial^m b^m (h-m)^{[k-m]}\,.$$

This is Lemma 3.8 from [3] written slightly differently.

Let $\binom{m}{i,j} = \frac{m!}{i!j!(m-i-j)!}$.

Proposition 3.5. *Let $\alpha = (a,b,c)$ be a multi-index, then*

$$D^\alpha\{s\} = \sum_{\substack{i+j\le a\\ k\le b}} \binom{a}{i,j}\binom{b}{k}\widehat{\partial}^{(a-i-j,b+i-k,c+j+k)} f_2^k f_1^i B^j (h_1-i-j)^{[a-i-j]}(h'\{s\}-j)^{[a-j]}(h_2\{s\}-j-k)^{[b-k]}\,.$$

Proof. It is not so difficult to proceed by induction on α with the help of the above lemmas. □

Corollary 3.6. $\ell ht\, D^\alpha\{s\} = \widehat{\partial}^\alpha h_1^{[\alpha_1]} h'\{s\}^{[\alpha_1]} h_2\{s\}^{[\alpha_2]}$.

The fact was proven in [3], but now it becomes just a corollary.

Proposition 3.7. *Let $\alpha = (a,b,c)$, and*

$$\alpha-(1) = (a-1,b,c)\,,\quad \alpha-(2) = (a,b-2,c)\,,\quad \alpha-(3) = (a,b,c-1)\,.$$

Then

$$\begin{aligned} e_0^\pm\, D^\alpha \;=\;& D^\alpha\{2\}\, e_0^\pm \;-a\, D^{\alpha-(1)}\{2\}\,\mathsf{d}_3^\pm B \;-b\, D^{\alpha-(2)}\{2\}\,\mathsf{d}_3^\pm f_2 \;-c\, D^{\alpha-(3)}\{1\}\,\mathsf{d}_3^\pm \\ &+\; ab\, D^{\alpha-(1)-(2)}\{2\}\,\mathsf{d}_3^\pm K\,, \end{aligned}$$

where $K = \widehat{\partial}_1 f_2 h' - \widehat{\partial}_2(f_{12}h_2 - f_1 f_2)$.

Let us mention that all terms on the right are shifted by 2 except $cD^{\alpha-(3)}\{1\}\mathsf{d}_3^\pm$, where the shift is indeed 1.

Proof. The proof goes through the repeated use of the Newton-Jacobson formula

$$x\, y^k \;=\; y^k\, x + \sum_{i=0}^{k-1}(-1)^i \binom{k}{i} y^{k-i}(\mathrm{ad} y)^i\, x$$

and a tedious calculation of commutators. It is advisable to write $D^\alpha = D_3^c D_2^b D_1^a$ making use of the commutativity of the operators. We leave the rest to the reader. □

Let $C = \widehat{\partial}_1 f_2 - \widehat{\partial}_2 f_{12}$. Then $K = Ch' + \widehat{\partial}_2 B$. Let us notice that

$$\begin{aligned} &C = [A, f_2] = [\,\widehat{\partial}_2, B\,]\,,\\ &[\,C, B\,] = [\,C, f_1] = [\,C, f_2] = [\,C, f_{12}] = 0\,,\\ &[\,C, \widehat{\partial}_i\,] = 0 \text{ for } i = 1,2,3\,. \end{aligned} \tag{3.4}$$

Define

$$K\{s\} = C(h'+s) + \widehat{\partial}_2 B\,. \tag{3.5}$$

Proposition 3.8.

$$f_2^k f_1^i B^j\, K = K\{j+k\} f_2^k f_1^i B^j - i\,\widehat{\partial}_1 f_{12}(h'+i-1-k) f_2^k f_1^{i-1} B^j\,.$$

Proof. It follows from (3.4) and the following lemma.

Lemma 3.9. (a): $[B,K]=CB, \quad BK\{s\}=K\{s+1\}B,$
(b): $[f_1,K]=-\widehat{\partial}_1 f_{12}h_1, \quad f_1^iK\{s\}=K\{s\}f_1^i-i\,\widehat{\partial}_1 f_{12}(h_1+i-1)f_1^{i-1},$
(c): $[f_2,K]=Cf_2, \quad f_2K\{s\}=K\{s+1\}f_2.$

We leave it to the reader to check the relations. □

Corollary 3.10. *In the notations of Proposition 3.7*

$$\ell ht\, D^{\alpha-(1)-(2)}\{2\}\,\mathsf{d}_3^{\pm}K=\widehat{\partial}^{\alpha-(2)}\mathsf{d}_3^{\pm}f_2h_1^{[a-1]}h'^{[a]}(h_2-1)^{[b-1]}\,.$$

Following Remark 2.3 and Proposition 3.2 we shall consider the $s\ell(3)$-highest vectors $w\in \mathsf{S}\,\Lambda^-\Lambda^+ V$ in the form

$$w=\sum_\alpha D^\alpha w_\alpha\,,\quad w_\alpha\in\Lambda^-\Lambda^+ V\,, \tag{3.6}$$

and assume that V is a submodule of either $\mathbb{C}[x_1,x_2,x_3]$ or $\mathbb{C}[\,\partial_1,\partial_2,\partial_3]$. We always assume that weights $\mathrm{wt}_2\,w$ and $\mathrm{wt}_3\,w$ are defined.

Let $N=\deg_{\mathsf{S}} w$, we shall call

$$w_{\text{top}}=\sum_{|\sigma|=N} D^\sigma w_\sigma\,, \tag{3.7}$$

the *top level* or the *top level terms* of w. We denote

$$w_{\text{top}-1}=\sum_{|\alpha|=N-1} D^\alpha w_\alpha\,, \tag{3.8}$$

and call it the *near-top level terms* of w. Our method to calculate w will be to determine its top level first, then its near-top level, and then the whole vector.

Proposition 3.11. *If $e_0^-w=0$, then in the notations of* (3.7) $e_0^-w_\sigma=0$ *for any* σ.

Proof. Let us use (3.6) and apply the formula of Proposition 3.7 in order to simplify $e_0^{\pm}\,w$. It follows from the relations

$$[e_0^-\,,\mathsf{d}_i^-]=0\,,\ \ [e_0^-\,,\mathsf{d}_1^+]=\ f_2\,,\ \ [e_0^-\,,\mathsf{d}_2^+]=-f_{12}\,,\ \ [e_0^-\,,\mathsf{d}_3^+]=0\,, \tag{3.9}$$

that $e_0^-\Lambda^-\Lambda^+ V\subset\Lambda^-\Lambda^+ V$. This helps to evaluate S-degrees of the various terms, and we conclude that

$$e_0^{\pm}\,w\equiv\sum_{|\sigma|=N} D\{2\}^\sigma e_0^{\pm}\,w_\sigma \quad \mod \deg_{\mathsf{S}}(<N)\,.$$

The statement follows with the help of Corollary 3.6. □

Remark 3.12. *In particular the proposition implies that if w is quasi-singular then w_σ are quasi-singular, and of course* $\mathrm{wt}_2\,w=\mathrm{wt}_2\,w_\sigma$.

We need to know explicitly the $sl(3)$-highest weight vectors in $\Lambda^- \Lambda^+ V$ for $V \subset \mathbb{C}[x_1, x_2, x_3]$, and $V \subset \mathbb{C}[\partial_1, \partial_2, \partial_3]$. To write them down let us introduce the following notations:

$$\begin{aligned}
\Delta^{\pm}(A_*) &:= \mathsf{d}_1^{\pm} A_1 + \mathsf{d}_2^{\pm} A_2 + \mathsf{d}_3^{\pm} A_3\,,\\
\widehat{\Delta}(A_*) &:= \widehat{\partial}_1 A_1 + \widehat{\partial}_2 A_2 + \widehat{\partial}_3 A_3\,,\\
(A_\circ B_\circ)_{i,j} &:= A_i B_j - A_j B_i\,,\\
\mathsf{C}(a_\bullet b_\bullet c_\bullet) &:= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2\,,\\
\Delta^{\pm}((A_\circ B_\circ)) &:= \mathsf{d}_1^{\pm}(A_2B_3 - A_3B_2) + \mathsf{d}_2^{\pm}(A_3B_1 - A_1B_3) + \mathsf{d}_3^{\pm}(A_1B_2 - A_2B_1)\,.
\end{aligned}$$

3(i). **Semi- and quasi-singular vectors for $V \cong F(p, 0)$.** We suppose that V is irreducible with the highest weight $(p, 0)$ throughout this subsection. In the following proposition we list the basis of the $s\ell(3)$-highest vectors w in $\Lambda^-\Lambda^+V$. We move from $\deg_\Lambda w = 0$ to $\deg_\Lambda w = 6$, and listing vectors with the same $\deg_\Lambda w$ and $\mathrm{wt}_3\, w$ we order them in a way that their $\mathrm{wt}_2\, w$ increases.

Proposition 3.13. *Let V be an irreducible $s\ell(3)$-submodule in $\mathbb{C}[x_1, x_2, x_3]$ generated by x_1^p. The $s\ell(3)$-highest vectors w in $\Lambda^-\Lambda^+V$ are linear combination of the following ones (we write $\mathrm{wt}_3\, w$ at the beginning of a line):*

$\deg_\Lambda w = 0$

$$(p+0,0):\ x_1^p\,,$$

$\deg_\Lambda w = 1$

$$\begin{aligned}
(p+1,0)&:\ \mathsf{d}_1^- x_1^p\,,\quad \mathsf{d}_1^+ x_1^p\,,\\
(p-1,1)&:\ (\mathsf{d}_\circ^- x_\circ)_{12}\, x_1^{p-1}\,,\quad (\mathsf{d}_\circ^+ x_\circ)_{12}\, x_1^{p-1}\,,
\end{aligned}$$

$\deg_\Lambda w = 2$

$$\begin{aligned}
(p+2,0)&:\ \mathsf{d}_1^-\mathsf{d}_1^+ x_1^p\,,\\
(p+0,1)&:\ \mathsf{d}_{12}^- x_1^p\,,\quad (\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\, x_1^p\,,\quad \mathsf{d}_1^-(\mathsf{d}_\circ^+ x_\circ)_{12}\, x_1^{p-1}\,,\quad \mathsf{d}_{12}^+ x_1^p\,,\\
(p-1,0)&:\ \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^- x_\bullet) x_1^{p-1}\,,\quad \Delta^-((\mathsf{d}_\circ^+ x_\circ)) x_1^{p-1}\,,\quad \mathsf{C}(\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+ x_\bullet) x_1^{p-1}\,,\\
(p-2,2)&:\ (\mathsf{d}_\circ^-(\mathsf{d}_\circ^+ x_\circ)_{12}\, x_\circ)_{12}\, x_1^{p-2}\,,
\end{aligned}$$

$\deg_\Lambda w = 3$

$$\begin{aligned}
(p+1,1)&:\ \mathsf{d}_{12}^-\mathsf{d}_1^+ x_1^p\,,\quad \mathsf{d}_1^-\mathsf{d}_{12}^+ x_1^p\,,\\
(p-1,2)&:\ \mathsf{d}_{12}^-(\mathsf{d}_\circ^+ x_\circ)_{12}\, x_1^{p-1}\,,\quad (\mathsf{d}_\circ^-\mathsf{d}_{12}^+ x_\circ)_{12}\, x_1^{p-1}\,,\\
(p+0,0)&:\ \mathsf{d}_{123}^- x_1^p\,,\quad \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+) x_1^p\,,\quad \mathsf{d}_1^-\Delta^-((\mathsf{d}_\circ^+ x_\circ)) x_1^{p-1}\,,\\
&\quad\ \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+) x_1^p\,,\quad \mathsf{d}_1^-\mathsf{C}(\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+ x_\bullet) x_1^{p-1}\,,\quad \mathsf{d}_{123}^+ x_1^p\,,\\
(p-2,1)&:\ \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-(\mathsf{d}_\circ^+ x_\circ)_{12}\, x_\bullet) x_1^{p-2}\,,\quad (\mathsf{d}_\circ^-\mathsf{C}(\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+ x_\bullet) x_\circ)_{12}\, x_1^{p-2}\,,
\end{aligned}$$

$\deg_\Lambda w = 4$

$(p+0,2): \ \mathsf{d}^-_{12}\mathsf{d}^+_{12}\, x_1^p\,,$

$(p+1,0): \ \mathsf{d}^-_{123}\mathsf{d}^+_1 x_1^p\,, \quad \mathsf{d}^-_1\,\mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^+_\bullet\mathsf{d}^+_\bullet)x_1^p\,, \quad \mathsf{d}^-_1\,\mathsf{d}^+_{123}\, x_1^p\,,$

$(p-1,1): \ \mathsf{d}^-_{123}(\mathsf{d}^+_\circ x_\circ)_{12}\, x_1^{p-1}\,, \quad \mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{d}^+_{12}\, x_\bullet)x_1^{p-1}\,,$

$\mathsf{d}^-_{12}\mathsf{C}(\mathsf{d}^+_\bullet\mathsf{d}^+_\bullet x_\bullet)x_1^{p-1}\,, \quad (\mathsf{d}^-_\circ\mathsf{d}^+_{123}\, x_\circ)_{12}\, x_1^{p-1}\,,$

$(p-2,0): \ \mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{C}(\mathsf{d}^+_\bullet\mathsf{d}^+_\bullet x_\bullet)x_\bullet)x_1^{p-1}\,,$

$\deg_\Lambda w = 5$

$(p+0,1): \ \mathsf{d}^-_{123}\mathsf{d}^+_{12}x_1^p\,, \quad \mathsf{d}^-_{12}\mathsf{d}^+_{123}\, x_1^p\,,$

$(p-1,0): \ \mathsf{d}^-_{123}\mathsf{C}(\mathsf{d}^+_\bullet\mathsf{d}^+_\bullet x_\bullet)x_1^{p-1}\,, \quad \mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{d}^+_{123}\, x_\bullet)x_1^{p-1}\,,$

$\deg_\Lambda w = 6$

$(p+0,0): \ \mathsf{d}^-_{123}\mathsf{d}^+_{123}\, x_1^p\,.$

Proof. The statement is a generalization of Lemma 3.12 from [3]. Its proof requires rather elementary but long calculations. We leave the details to the reader. □

Corollary 3.14. *Admissible (i.e. of non-negative $s\ell(2)$-weight) quasi-singular vectors in $\Lambda^-\Lambda^+V$ are:*

$p \geq 0: \quad x_1^p\,, \quad \mathsf{d}^+_1\, x_1^p\,, \quad \mathsf{d}^-_1\mathsf{d}^+_1\, x_1^p\,, \quad (\mathsf{d}^-_\circ\mathsf{d}^+_\circ)_{12}\, x_1^p + p(\mathsf{d}^-_\circ\mathsf{d}^+_1 x_\circ)_{12}\, x_1^{p-1}\,,$

and also:

$p = 0: \quad \mathsf{d}^+_{123}\,, \quad \mathsf{d}^-_1\mathsf{d}^+_{123}\,, \quad \mathsf{d}^-_{12}\mathsf{d}^+_{123}\,, \quad \mathsf{d}^-_{123}\mathsf{d}^+_{123}\,.$

Proof. We need to apply e_0^- to each of the expressions of the proposition with non-negative $s\ell(2)$-weight and collect the cases when it gives zero. It could be done easily using the relations (3.9). Let us mention that later we shall utilize the results for the cases when they are not zero. □

This gives us the description of admissible quasi-singular vectors $w \in \mathsf{S}\Lambda^-\Lambda^+V$ such that $\deg_{\mathsf{S}} w = 0$. To describe the vectors with $\deg_{\mathsf{S}} w > 0$ is undoubtedly more complicated, and the answer is given by the theorems below. We present the description of quasi-singular vectors first, then we use it to find semi-singular vectors.

Theorem 3.15. *Let V be an irreducible $s\ell(3)$-submodule in $\mathbb{C}[x_1,x_2,x_3]$ generated by x_1^p. The following is a complete list of admissible quasi-singular vectors w in $\mathsf{S}\Lambda^-\Lambda^+V$ such that* $\mathrm{wt}_2\, w \geq 0$ *and* $\deg_{\mathsf{S}} w > 0$*, up to taking linear cominations :*

(1) $(D_1^n\mathsf{d}^-_1\mathsf{d}^+_1 - n(p+3)D_1^{n-1}\mathsf{d}^-_1(\mathsf{d}^-_\circ\mathsf{d}^+_\circ)_{23}\mathsf{d}^+_1 - n^{[2]}(p+3)^{[2]}D_1^{n-2}\mathsf{d}^-_{123}\mathsf{d}^+_{123})x_1^p\,,$
$n \leq p+2\,,$

(2) $D_3^n\mathsf{d}^-_{123}\mathsf{d}^+_{123}\,,\ n \geq 0\,,\ p = 0\,,$

(3) $\widehat{\Delta}(\mathsf{d}_*^+)\widehat{\Delta}(x_*) - \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)\widehat{\Delta}(x_*) + \widehat{\Delta}(\mathsf{d}_*^-)\mathsf{C}(\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+x_\bullet) + \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_{123}^+\,x_\bullet)\,,\ p=1\,,$

(4) $D_2\Big((\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\,x_1^p + p(\mathsf{d}_\circ^-\mathsf{d}_1^+x_\circ)_{12}\,x_1^{p-1}\Big) - \mathsf{d}_1^-(\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{23}\,\mathsf{d}_1^+x_1^p\,,\ p\geq 0\,,$

(5) $w = \widehat{\Delta}(x_*) - \frac{1}{2}\Delta^-((\mathsf{d}_\circ^+x_\circ))\,,\ p=1\,,$

(6) $\frac{1}{3}D_1\mathsf{d}_1^+x_1 - \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)x_1 + \mathsf{d}_1^-\mathsf{C}(\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+x_\bullet)\,,\ p=1\,,$

(7) $D_1\big(\mathsf{d}_1^-f_1(\mathsf{d}_1^+x_1) - 2\mathsf{d}_2^-\mathsf{d}_1^+x_1\big) - 3\,\big(\mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_{12}^+\,x_\bullet) + \mathsf{d}_{12}^-\mathsf{C}(\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+x_\bullet)\big)\,,\ p=1\,,$

(8) $\widehat{\Delta}(\mathsf{d}_*^+) - \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)\,,\ p=0\,,$

(9) $\widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_{123}^+\,,\ p=0\,,$

(10) $(\widehat{\partial}_2\mathsf{d}_{12}^- + \widehat{\partial}_3\mathsf{d}_{13}^-)\mathsf{d}_{123}^+\,,\ p=0\,.$

Let us notice that only in (1) p and $\deg_{\mathsf{S}} w$ are unbounded, for all cases, except (1) and (2), we have $\deg_{\mathsf{S}} w \leq 2$, and for all cases, except (1) and (4), $p \leq 1$.

We shall check first that all these vectors are indeed quasi-singular. While checking (1) we use Proposition 3.7. To check the rest becomes elementary calculation based on the relations (3.9) and the fact that

$$[e_0^{\pm}\,, \widehat{\partial}_i] = 0\,. \tag{3.10}$$

Theorem 3.16. *Let V be an irreducible $s\ell(3)$-submodule in $\mathbb{C}[x_1,x_2,x_3]$ generated by x_1^p. The admissible semi-singular vectors w in $\mathsf{S}\Lambda^-\Lambda^+V$ such that* $\mathrm{wt}_2\, w \geq 0$ *and* $\deg_{\mathsf{S}} w > 0$ *exist only when $p=0$ and they are the following:*

(i) $\widehat{\Delta}(\mathsf{d}_*^+) - \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)\,,$

(ii) $\widehat{\Delta}(\mathsf{d}_*^-)\,(\widehat{\Delta}(\mathsf{d}_*^+) - \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+))\,,$

(iii) $\mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^+) - \mathsf{d}_1^-(\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{23}\mathsf{d}_1^+ = \mathsf{d}_1^-(\widehat{\Delta}(\mathsf{d}_*^+) - \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+))\,,$

(iv) $\widehat{\Delta}(\mathsf{d}_*^-)\,\mathsf{d}_{123}^+\,,$

(v) $(\widehat{\partial}_2\mathsf{d}_{12}^- + \widehat{\partial}_3\mathsf{d}_{13}^-)\mathsf{d}_{123}^+ = \mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^-)\,\mathsf{d}_{123}^+\,.$

We also need to check that vectors listed in Theorem 3.16 are semi-singular. Here the calculations use the relations of commutation with e_0^+:

$$[e_0^+\,,\mathsf{d}_1^-] = -\,f_2\,,\ [e_0^+\,,\mathsf{d}_2^-] = f_{12}\,,\ [e_0^+\,,\mathsf{d}_3^-] = 0\,.\ [e_0^+\,,\mathsf{d}_i^+] = 0\,, \tag{3.11}$$

and are pretty straightforward.

What left is the hard parts of the theorems, the statements that the lists contain all the vectors. The proof of this demands to work through an elaborated "tree of cases" and will be postponed to the next section.

3(ii). **Semi- and quasi-singular vectors for $V \cong F(0,q)$.** We suppose that V is irreducible with the highest weight $(0,q)$, $q \geq 1$ throughout this subsection.

Similarly we need first of all the description of the $s\ell(3)$-highest vectors w in $\Lambda^-\Lambda^+V$.

Proposition 3.17. *Let V be an irreducible $s\ell(3)$-submodule in $\mathbb{C}[\partial_1,\partial_2,\partial_3]$ generated by ∂_3^q. The $s\ell(3)$-highest vectors w in $\Lambda^-\Lambda^+V$ are linear combination of the following ones:*

$\deg_\Lambda w = 0$

$$(0,q+0): \ \partial_3^q\,,$$

$\deg_\Lambda w = 1$

$$\begin{aligned}
&(0,q-1): \ \Delta^-(\partial_*)\partial_3^{q-1}\,,\quad \Delta^+(\partial_*)\partial_3^{q-1}\,,\\
&(1,q+0): \ \mathsf{d}_1^-\partial_3^q\,,\quad \mathsf{d}_1^+\partial_3^q\,,
\end{aligned}$$

$\deg_\Lambda w = 2$

$$\begin{aligned}
&(0,q-2): \ \Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}\,,\\
&(1,q-1): \ \mathsf{d}_1^-\Delta^-(\partial_*)\partial_3^{q-1}\,,\quad \mathsf{d}_1^-\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \Delta^-(\mathsf{d}_1^+\partial_*)\partial_3^{q-1}\,,\quad \mathsf{d}_1^+\Delta^+(\partial_*)\partial_3^{q-1}\,,\\
&(0,q+1): \ \mathsf{d}_{12}^-\partial_3^q\,,\quad (\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\partial_3^q\,,\quad \mathsf{d}_{12}^+\partial_3^q\,,\\
&(2,q+0): \ \mathsf{d}_1^-\mathsf{d}_1^+\partial_3^q\,,
\end{aligned}$$

$\deg_\Lambda w = 3$

$$\begin{aligned}
&(1,q-2): \ \mathsf{d}_1^-\Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}\,,\quad \Delta^-(\mathsf{d}_1^+\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}\,,\\
&(0,q+0): \ \mathsf{d}_{123}^-\partial_3^q\,,\quad \mathsf{d}_{12}^-\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+)\partial_3^q\,,\\
&\qquad\qquad (\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)\partial_3^q\,,\quad \mathsf{d}_{123}^+\partial_3^q\,,\\
&(2,q-1): \ \mathsf{d}_1^-\Delta^-(\mathsf{d}_1^+\partial_*)\partial_3^{q-1}\,,\quad \mathsf{d}_1^-\mathsf{d}_1^+\Delta^+(\partial_*)\partial_3^{q-1}\,,\\
&(1,q+1): \ \mathsf{d}_{12}^-\mathsf{d}_1^+\partial_3^q\,,\quad \mathsf{d}_1^-\mathsf{d}_{12}^+\partial_3^q\,,
\end{aligned}$$

$\deg_\Lambda w = 4$

$$\begin{aligned}
&(0,q-1): \ \mathsf{d}_{123}^-\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+)\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \Delta^-(\mathsf{d}_{123}^+\partial_*)\partial_3^{q-1}\,,\\
&(2,q-2): \ \mathsf{d}_1^-\Delta^-(\mathsf{d}_1^+\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}\,,\\
&(1,q+0): \ \mathsf{d}_{123}^-\mathsf{d}_1^+\partial_3^q\,,\quad \mathsf{d}_{12}^-\mathsf{d}_1^+\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \mathsf{d}_1^-\mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)\partial_3^q\,,\quad \mathsf{d}_1^-\mathsf{d}_{123}^+\partial_3^q\,,\\
&(0,q+2): \ \mathsf{d}_{12}^-\mathsf{d}_{12}^+\partial_3^q\,,
\end{aligned}$$

$\deg_\Lambda w = 5$

$(1,q-1):\ \mathsf{d}^-_{123}\mathsf{d}^+_1\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \mathsf{d}^-_1\Delta^-(\mathsf{d}^+_{123}\partial_*)\partial_3^{q-1}\,,$

$(0,q+1):\ \mathsf{d}^-_{123}\mathsf{d}^+_{12}\partial_3^{q}\,,\quad \mathsf{d}^-_{12}\mathsf{d}^+_{123}\partial_3^{q}\,,$

$\deg_\Lambda w = 6$

$(0,q+0):\ \mathsf{d}^-_{123}\mathsf{d}^+_{123}\partial_3^{q}\,.$

Proof. This is also a generalization of Lemma 3.12 from [3] and we leave it as an exercise to the reader. □

Corollary 3.18. *Admissible quasi-singular vectors in $\Lambda^-\Lambda^+V$ are:*

$$\partial_3^q\,,\quad \Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \mathsf{d}^-_1\Delta^+(\partial_*)\partial_3^{q-1}\,,\quad \textit{where } q\ge 1\,,\quad \Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}\,,\quad q\ge 2\,,$$

and more for $q=1$:

$$\mathsf{d}^+_1\Delta^+(\partial_*)\,,\quad (\mathsf{d}^-_\circ\mathsf{d}^+_\circ)_{12}\Delta^+(\partial_*)\,,\quad \mathsf{d}^-_1\mathsf{d}^+_1\Delta^+(\partial_*)\,,\quad \mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{d}^+_\bullet)\Delta^+(\partial_*)\,,\quad \mathsf{d}^-_{12}\mathsf{d}^+_1\Delta^+(\partial_*)\,.$$

We need only to calculate the action of e_0^- on the vectors of non-negative $s\ell(2)$-weight of the proposition. Corollary collects the cases when the results are zero. The results for the cases when the results are non-zero will be needed later too.

Admissible semi-singular and quasi-singular vectors w with $\deg_{\mathsf{S}} w\ge 0$ are described by the following theorems.

Theorem 3.19. *Let V be an irreducible $s\ell(3)$-submodule in $\mathbb{C}[\partial_1,\partial_2,\partial_3]$ generated by ∂_3^q, $q\ge 1$. The admissible quasi-singular vectors w in $\mathsf{S}\Lambda^-\Lambda^+V$ such that* $\mathrm{wt}_2\, w\ge 0$ *and* $\deg_{\mathsf{S}} w>0$ *are given by the following expressions:*

(1) $D_3^n\Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}+\frac{n}{q-1}D_3^{n-1}\mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{d}^+_\bullet)\Delta^+(\partial_*)\partial_3^{q-1}-\frac{n^{[2]}}{q^{[2]}}D_3^{n-2}\mathsf{d}^-_{123}\mathsf{d}^+_{123}\partial_3^{q}\,,$ $n\ge 1,\ q\ge 2\,,$

(2) $D_1\,\mathsf{d}^-_1\Delta^+(\partial_*)\partial_3^{q-1}+\mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{d}^+_\bullet)\Delta^+(\partial_*)\partial_3^{q-1}\,,$

(3) $D_1D_2\,\mathsf{d}^-_1\Delta^+(\partial_*)\partial_3-\frac{1}{2}D_1\mathsf{d}^-_1\Delta^-(\mathsf{d}^+_1\Delta^+(\partial_*)\partial_*)-\frac{3}{2}D_2\mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{d}^+_\bullet)\Delta^+(\partial_*)\partial_3\,,\ q=2\,,$

(4) $D_2\,\mathsf{d}^-_1\Delta^+(\partial_*)\partial_3-\mathsf{d}^-_1\Delta^-(\mathsf{d}^+_1\Delta^+(\partial_*)\partial_*)\,,\ q=2\,,$

(5) $D_2\partial_3-\Delta^-(\mathsf{d}^+_1\partial_*)\,,\ q=1\,,$

(6) $D_1\mathsf{d}^+_1\Delta^+(\partial_*)-2\Delta^-(\mathsf{d}^+_{123}\partial_*)\,,\ q=1\,,$

(7) $D_2(\mathsf{d}^-_\circ\mathsf{d}^+_\circ)_{12}\Delta^+(\partial_*)-\mathsf{d}^-_1\Delta^-(\mathsf{d}^+_{123}\partial_*)\,,\ q=1\,,$

(8) $D_3^n\mathsf{C}(\mathsf{d}^-_\bullet\mathsf{d}^-_\bullet\mathsf{d}^+_\bullet)\Delta^+(\partial_*)-nD_3^{n-1}\mathsf{d}^-_{123}\mathsf{d}^+_{123}\partial_3\,,\ n\ge 1\,,\ q=1\,,$

(9) $D_1\mathsf{d}^-_{12}\mathsf{d}^+_1\Delta^+(\partial_*)-2\,\mathsf{d}^-_{123}\mathsf{d}^+_{123}\partial_3\,,\ q=1\,,$

(10) $D_1^n\,\mathsf{d}_1^-\mathsf{d}_1^+\Delta^+(\partial_*) - 3nD_1^{n-1}\mathsf{d}_1^-\Delta^-(\mathsf{d}_{123}^+\partial_*),\ n=1,2\,,\ q=1\,.$

This is a complete list of such vectors up to taking linear combinations.

It is not difficult to check that these vectors are indeed quasi-singular. For example for the case (3), after making calculations according to Proposition 3.7 we get

$$\begin{aligned} e_0^- D_1 D_2\,\mathsf{d}_1^-\Delta^+(\partial_*)\partial_3 &= +D_2\{2\}\mathsf{d}_3^-\Delta^-(\Delta^+(\partial_*)\partial_*) \\ &\quad +D_1\{2\}\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_2 \\ &\quad +\widehat{\partial}_1(-3\,\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_2) + \widehat{\partial}_2(\,2\,\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_1 - \mathsf{d}_{32}^-\Delta^+(\partial_*)\partial_2)\,. \end{aligned}$$

At the same time

$$\begin{aligned} D_1\{2\}\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_2 &= \widehat{\partial}_1(+6\,\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_2)\ + \widehat{\partial}_2(-3\,\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_1 + 3\,\mathsf{d}_{32}^-\Delta^+(\partial_*)\partial_2)\,, \\ D_2\{2\}\mathsf{d}_3^-\Delta^-(\Delta^+(\partial_*)\partial_*) &= \widehat{\partial}_2(+\,\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_1 + \mathsf{d}_{32}^-\Delta^+(\partial_*)\partial_2)\,. \end{aligned}$$

Also

$$e_0^- D_1\mathsf{d}_1^-\Delta^-(\mathsf{d}_1^+\Delta^+(\partial_*)\partial_*) = D_1\{2\}\mathsf{d}_{31}^-\Delta^+(\partial_*)\partial_2 - 3\,\mathsf{d}_{312}^-\mathsf{d}_3^+\Delta^+(\partial_*)\partial_2\,,$$

and

$$e_0^- D_2\mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+)\Delta^+(\partial_*)\partial_3 = D_2\{2\}\mathsf{d}_3^-\Delta^-(\Delta^+(\partial_*)\partial_*) + \mathsf{d}_{312}^-\mathsf{d}_3^+\Delta^+(\partial_*)\partial_2\,.$$

Therefore we conclude that

$$e_0^-\left(D_1D_2\,\mathsf{d}_1^-\Delta^+(\partial_*)\partial_3 - \tfrac{1}{2}D_1\mathsf{d}_1^-\Delta^-(\mathsf{d}_1^+\Delta^+(\partial_*)\partial_*) - \tfrac{3}{2}D_2\mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+)\Delta^+(\partial_*)\partial_3\right) = 0\,.$$

We leave to check the rest of vectors to the reader. The proof that there are no other quasi-singular vectors will be presented later in a separate section.

Theorem 3.20. *Let V be irreducible $s\ell(3)$-submodule in $\mathbb{C}[\partial_1,\partial_2,\partial_3]$ generated by ∂_3^q, $q\geq 1$. The admissible semi-singular vectors w in $\mathsf{S}\Lambda^-\Lambda^+V$ such that* $\mathrm{wt}_2\,w\geq 0$ *and* $\deg_{\mathsf{S}} w>0$ *exist only for $q=1$ and $q=2$. They are the following vectors:*

(i) $\widehat{\partial}_2\partial_3 - \widehat{\partial}_3\partial_2 - \Delta^-(\mathsf{d}_1^+\partial_*)\,,$

(ii) $\widehat{\partial}_2\mathsf{d}_1^-\Delta^+(\partial_*)\partial_3 - \widehat{\partial}_3\mathsf{d}_1^-\Delta^+(\partial_*)\partial_2 - \mathsf{d}_1^-\Delta^-(\mathsf{d}_1^+\Delta^+(\partial_*)\partial_*)\,,$

(iii) $(\widehat{\Delta}(\mathsf{d}_*^-) - \mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+))\Delta^+(\partial_*)\,,$

(iv) $(\widehat{\partial}_1\mathsf{d}_1^+ + \widehat{\partial}_2\mathsf{d}_2^+ + \widehat{\partial}_3\mathsf{d}_3^+)\Delta^+(\partial_*) - \Delta^-(\mathsf{d}_{123}^+\partial_*)\,,$

(v) $(\widehat{\partial}_1\mathsf{d}_1^-\mathsf{d}_1^+ + \widehat{\partial}_2\mathsf{d}_1^-\mathsf{d}_2^+ + \widehat{\partial}_3\mathsf{d}_1^-\mathsf{d}_3^+)\Delta^+(\partial_*) - \mathsf{d}_1^-\Delta^-(\mathsf{d}_{123}^+\partial_*)\,,$

(vi) $D_1^2\,\mathsf{d}_1^-\mathsf{d}_1^+\Delta^+(\partial_*) - 6\,D_1\,\mathsf{d}_1^-\Delta^-(\mathsf{d}_{123}^+\partial_*)\,.$

It is not so difficult to check that these vectors are semi-singular. The proof that Theorems 3.16, 3.20 describe all such vectors will be discussed later in a separate section.

Corollary 3.21. *Admissible semi-singular vectors with* $\deg_{\mathsf{S}} w = 0$ *in* $\Lambda^-\Lambda^+V$ *are:*

$$\partial_3^q\,,\quad \overset{+}{\Delta}(\partial_*)\partial_3^{q-1}\,,\qquad \textit{where } q \geq 1\,,\quad \overset{-}{\Delta}(\overset{+}{\Delta}(\partial_*)\partial_*)\,,\quad q=2\,,$$

$$\mathsf{d}_1^+\overset{+}{\Delta}(\partial_*)\,,\quad \mathsf{d}_1^-\overset{+}{\Delta}(\partial_*)\,,\quad \mathsf{d}_1^-\mathsf{d}_1^+\overset{+}{\Delta}(\partial_*)\,,\quad q=1\,.$$

One easily gets the list from Corollary 3.18 applying e_0^+. We shall use it in Section 8.

4. **Proof of Theorem 3.15.**

We consider a quasi-singular vector $w \in \mathsf{S}\Lambda^-\Lambda^+V$ under the conditions of Theorem 3.15. Without loss of generality we can assume that w is a $g\ell(1)$-weight vector. We keep the notations of (3.6) and (3.7).

First of all let us notice that for $|\alpha| = N - i$

$$\deg_\Lambda w_\alpha = \deg_\Lambda w_\varkappa + 2i\,,\quad \mathrm{wt}_2\, w_\alpha = \mathrm{wt}_2\, w_\varkappa = \mathrm{wt}_2\, w\,, \tag{4.1}$$

because $\mathrm{wt}_2\, w = \mathrm{wt}_2\, D^\alpha w_\alpha = \mathrm{wt}_2\, w_\alpha$, and $\mathrm{wt}_1\, w = \mathrm{wt}_1\, D^\alpha w_\alpha = -\frac{2}{3}|\alpha| + \mathrm{wt}_1\, w_\alpha$, but wt_1 of an element in $\Lambda^-\Lambda^+V$ depends only on its Λ-degree.

Suppose that $\varkappa$ is the lexicographically highest element in

$$I_{\mathrm{top}} := \{\sigma \mid w_\sigma \neq 0\,,\quad \text{where } |\sigma| = N = \deg_{\mathsf{S}} w\,\}\,,$$

(i.e. w_σ are the top level coefficients of w). It is clear that in order to prove that a quasi-singular vector is a linear combination of vectors listed in the statement of the theorem, it is enough to show that the list contains a vector for each possible "highest term" $w_\varkappa$, then the result follows by induction. Therefore it is enough to prove that any possible "highest term" $w_\varkappa$ is present among the "highest terms" $w_\varkappa$ of the vectors listed in Theorem 3.15.

Remark 4.1. *It follows from* (4.1) *that* $\deg_\Lambda w_\sigma = \deg_\Lambda w_\varkappa$ *and* $\mathrm{wt}_2\, w_\sigma = \mathrm{wt}_2\, w_\varkappa$. *These are important restrictions on* w_σ.

Proposition 3.11 and Remark 3.12 show that w_σ are quasi-singular vectors of non-negative $s\ell(2)$-weight, thus belong to the list given by Corollary 3.14.

Proposition 4.2. (1): *If* $w_\varkappa = x_1^p\,,\quad \mathsf{d}_1^- x_1^p$, *or* $(\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\, x_1^p + p(\mathsf{d}_\circ^-\mathsf{d}_1^+ x_\circ)_{12}\, x_1^{p-1}\,,$ *then* $I_{top} = \{\varkappa\}$.
(2): *If* $w_\varkappa = \mathsf{d}_1^-\mathsf{d}_1^+ x_1^p$, *then either* $I_{top} = \{\varkappa\}$ *or* $I_{top} = \{\varkappa, \sigma\}$, *where* $\sigma = \varkappa - (1,-1,0)$ *and* $w_\sigma = c_\sigma((\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\, x_1^p + p(\mathsf{d}_\circ^-\mathsf{d}_1^+ x_\circ)_{12}\, x_1^{p-1})\,,\; c_\sigma \in \mathbb{C}$.
(3): *If* $w_\varkappa = \mathsf{d}_{123}^+\,,\quad \mathsf{d}_1^-\mathsf{d}_{123}^+\,,\quad \mathsf{d}_{12}^-\mathsf{d}_{123}^+\,,$ *or* $\mathsf{d}_{123}^-\mathsf{d}_{123}^+\,,$ *then* $I_{top} = \{\varkappa\}$.

Proof. The result follows from Remark 4.1, Remark 3.12 and the fact that $\mathrm{wt}_3\, D^\sigma w_\sigma = \mathrm{wt}_3\, D^\varkappa w_\varkappa$ as soon as one checks the weights wt_1 , wt_2 , wt_3 through the list of Corollary 3.14. □

The proposition describes possibilities for w_{top}, now we are to determine what is possible for $w_{\mathrm{top}-1}$. We are interested only in terms $D^\alpha w_\alpha \neq 0$ with $|\alpha| = N - 1$ and $e_0^- w_\alpha \neq 0$. The following will be of use.

Remark 4.3. *Let $D^\alpha w_\alpha \neq 0$, and consider any β, such that $0 \leq \beta_i \leq \alpha_i$. Then $D^\beta w_\alpha \neq 0$, because $D^\alpha w_\alpha = D^{\alpha-\beta} D^\beta w_\alpha$, hence the weight*

$$\mathrm{wt}_3\, D^\beta w_\alpha = \mathrm{wt}_3\, w_\alpha + \beta_1(-1,0) + \beta_2(1,-1) + \beta_3(0,1)$$

is dominant by Proposition 3.2. For example one can take $\beta = (\alpha_1, 0, 0)$ or $\beta = (0, \alpha_2, 0)$.

By Proposition 3.7 the condition $e_0^- w = 0$ implies

$$\sum_{|\alpha|=N-1} D\{2\}^\alpha e_0^- w_\alpha \equiv -e_0^- w_{\mathrm{top}} \mod \deg_{\mathsf{S}}(< N-1)\,. \tag{4.2}$$

Applying again Proposition 3.7 we conclude that

$$\sum_{|\alpha|=N-1} D\{2\}^\alpha e_0^- w_\alpha \equiv \varkappa_3\, D^{\varkappa-(3)}\{1\}\, \mathsf{d}_3^- w_\varkappa \mod \mathbf{lex}(<\ \varkappa_{-(3)})\,, \tag{4.3}$$

because all the other terms in $e_0^- w_{\mathrm{top}}$ are lexicographically smaller. From now on we study the possibilities of Proposition 4.2 **case by case**.

Case 1(i): $\underline{w_\varkappa = x_1^p}$.
Here Remark 4.3 shows that $\varkappa_1 \leq p$ and $\varkappa_2 = 0$.

Lemma 4.4. *$\varkappa_3 = 0$.*

Proof. In our case $w_{\mathrm{top}} = D^\varkappa x_1^p$. From (4.3) it follows that either $\varkappa_3 = 0$ or $e_0^- w_{\varkappa-(3)} \neq 0$. We shall show that the latter is impossible.

We are to look for the values of $w_{\varkappa-(3)}$. First of all, from (4.1) we see that $\mathrm{wt}_2\, w_{\varkappa-(3)} = 0$, $\deg_\Lambda w_{\varkappa-(3)} = 2$, thus $w_{\varkappa-(3)} \in \Lambda_1^- \Lambda_1^+ V$. As $\mathrm{wt}_3\, D^\varkappa w_\varkappa = \mathrm{wt}_3\, D^\alpha w_\alpha$, it is clear that $\mathrm{wt}_3\, w_{\varkappa-(3)} = (p, 1)$, and the list of Proposition 3.13 provides us with the only choice

$$w_{\varkappa-(3)} = c_1 (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{12}\, x_1^p + c_2\, \mathsf{d}_1^- (\mathsf{d}_\circ^+ x_\circ)_{12}\, x_1^{p-1}\,.$$

Then $e_0^- w_{\varkappa-(3)} = \mathsf{d}_1^- A + \mathsf{d}_2^- B$ for some $A, B \in \Lambda^+ V$ because $[\, e_0^-, \mathsf{d}_i^- \,] = 0$, and the equality (4.3) is impossible when $e_0^- w_{\varkappa-(3)} \neq 0$. ☐

We conclude that $\varkappa = (N, 0, 0)$, and (4.2), (4.3) give us

$$\sum_{|\alpha|=N-1} D\{2\}^\alpha e_0^- w_\alpha \equiv \varkappa_1\, D^{\varkappa-(1)}\{2\}\, \mathsf{d}_3^- B\, x_1^p \mod \mathbf{lex}(<\ \varkappa_{-(1)})\,, \tag{4.4}$$

where $B\, x_1^p = (p+1)\, x_3 x_1^{p-1}$.

Lemma 4.5. *$N = 1$, $p = 1$.*

Proof. By the same arguments as above $w_{\varkappa-(1)} \in \Lambda_1^- \Lambda_1^+ V$, $\mathrm{wt}_2\, w_{\varkappa-(1)} = 0$, and it is easy to see that $\mathrm{wt}_3\, w_{\varkappa-(1)} = (p-1, 0, 0)$. Then from Proposition 3.13 it follows that $w_{\varkappa-(1)} = c\, \Delta^- ((\mathsf{d}_\circ^+ x_\circ)) x_1^{p-1}$. We calculate that

$$e_0^- w_{\varkappa-(1)} = c \left((p-1) \mathsf{d}_1^- x_3^2 x_1^{p-2} - (p+1) \mathsf{d}_3^- x_3 x_1^{p-1} \right)\,.$$

Clearly (4.4) implies that $p = 1$. But $N = \varkappa_1 \leq p$ in our case, and because $N = \deg_{\mathsf{S}} w > 0$ we get $N = 1$. □

As a result we have come to the vector given in Theorem 3.15(5).

Case 1(ii): $\underline{w_\varkappa = \mathsf{d}_1^+ x_1^p}$.
We follow the same line of arguments. By Remark 4.3 $\varkappa_1 \leq p+1$ and $\varkappa_2 = 0$ in this case.

Lemma 4.6. $\varkappa_3 = 0$.

Proof. Consider the terms $D^\alpha w_\alpha$ with $|\alpha| = N-1$ and $e_0^- w_\alpha \neq 0$. Here we immediately get $\mathrm{wt}_2\, w_\alpha = +1$ and $w_\alpha \in \Lambda_1^- \Lambda_2^+ V$. Now from (4.3) it follows that if $\varkappa_3 \neq 0$, then $e_0^- w_{\varkappa-(3)} \neq 0$, and we are to look for the value of $w_{\varkappa-(3)}$ in the list of Proposition 3.17. In this case $\mathrm{wt}_3\, w_{\varkappa-(3)} = (p+1, 1, 0)$ and we get $w_{\varkappa-(3)} = c\, \mathsf{d}_1^- \mathsf{d}_{12}^+ x_1^p$. But this does not fit into (4.3). □

We come to $\varkappa = (N, 0, 0)$, and to the equation

$$\sum_{|\alpha|=N-1} D\{2\}^\alpha e_0^- w_\alpha \equiv \varkappa_1\, D^{\varkappa-(1)}\{2\}\, \mathsf{d}_3^- B\, \mathsf{d}_1^+ x_1^p \quad \mathrm{mod}\ \mathbf{lex}(<\ \varkappa_{-(1)})\,, \tag{4.5}$$

where $B\, \mathsf{d}_1^+ x_1^p = (p+2)\, f_{12} \mathsf{d}_1^+ x_1^{p-1}$. We also see that $\mathrm{wt}_3\, w_{\varkappa-(1)} = (p, 0, 0)$. Then from Proposition 3.13 it follows that $w_{\varkappa-(1)} = c_1\, \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^+ \mathsf{d}_\bullet^+) x_1^p + c_2\, \mathsf{d}_1^- \mathsf{C}(\mathsf{d}_\bullet^+ \mathsf{d}_\bullet^+ x_\bullet) x_1^{p-1}$.
Hence

$$\begin{aligned} e_0^- w_{\varkappa-(1)} = \quad & c_1\, (p\, \mathsf{d}_1^- \mathsf{d}_3^+ x_3 x_1^{p-1} - \mathsf{d}_3^- \mathsf{d}_3^+ x_1^p - p \mathsf{d}_3^- \mathsf{d}_1^+ x_3 x_1^{p-1}) \\ + \quad & c_2\, (p\, \mathsf{d}_1^- \mathsf{d}_3^+ x_3 x_1^{p-1} - (p-1)\, \mathsf{d}_1^- \mathsf{d}_3^+ x_3^2 x_1^{p-2})\,. \end{aligned}$$

Clearly $p \leq 1$ and either $p = 1$, $c_1 + c_2 = 0$, or $p = 0$, $c_2 = 0$ could satisfy (4.5). Now $p = 1$, $N = 1$ gives us Theorem 3.15(6), $p = 1$, $N = 2$ gives Theorem 3.15(3) and $p = 0$, $N = 1$ gives Theorem 3.15(8).

Case 1(iii): $\underline{w_\varkappa = (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{12}\, x_1^p + p(\mathsf{d}_\circ^- \mathsf{d}_1^+ x_\circ)_{12}\, x_1^{p-1}}$.
By Remark 4.3 we see that $\varkappa_1 \leq p$ and $\varkappa_2 \leq 1$.

Lemma 4.7. $\varkappa_3 = 0$.

Proof. Here $\mathrm{wt}_2\, w_\alpha = 0$, $\deg_\Lambda w_\alpha = 4$ for $|\alpha| = N-1$, hence $w_\alpha \in \Lambda_2^- \Lambda_2^+ V$.

For $\alpha = \varkappa_{-(3)}$ we have $\mathrm{wt}_3\, w_{\varkappa-(3)} = \mathrm{wt}_3\, w_\alpha + (0,1) = (p, 2)$. Therefore we find with the help of Proposition 3.13 that $w_{\varkappa-(3)} = c\, \mathsf{d}_{12}^- \mathsf{d}_{12}^+ x_1^p$. But from (4.3) it follows that

$$\sum_{|\alpha|=N-1} D\{2\}^\alpha e_0^- w_\alpha \equiv \varkappa_3\, D^{\varkappa-(3)}\{1\}\, \mathsf{d}_3^- \mathsf{d}_1^- w_\varkappa \quad \mathrm{mod}\ \mathbf{lex}(<\ \varkappa_{-(3)}), \tag{4.6}$$

which is only possible with the above value of $w_{\varkappa-(3)}$ when $\varkappa_3 = 0$, $c = 0$. □

Let us remind that $\varkappa_2 \leq 1$.

Subcase: $\varkappa_2 = 1$.
Let $\varkappa = (n,1,0)$. Now from Proposition 3.7 we get

$$e_0^- D^{\varkappa} w_{\varkappa} = D_1^{n-1}\{2\}\left(-n\, D_2\{2\}\mathsf{d}_3^- B\, w_{\varkappa} - D_1\{2\}\mathsf{d}_3^- f_2 w_{\varkappa} + n\,\mathsf{d}_3^- K\, w_{\varkappa}\right) . \tag{4.7}$$

This and (4.2) implies

$$D_1^{n-1}\{2\}\left(D_2\{2\}\mathsf{d}_3^- B + n(D_1\{2\}\mathsf{d}_3^- f_2 - \mathsf{d}_3^- K)\right) w_{\varkappa} \equiv \sum_{|\alpha|=n} D\{2\}^{\alpha} e_0^- w_{\alpha} \bmod \deg_{\mathsf{S}}(< n)\,,$$

where we will ignore those α where $e_0^- w_{\alpha} = 0$.

Let us consider the weaker congruence, the same expression but

$$\bmod \mathbf{dlex}(< (n-1,1,0)),$$

i.e. we use the degree-lexicographic order defined in (3.2). Then only $\alpha = (n,0,0)$ and $\alpha = (n-1,1,0)$ are to be considered, and we come to an expression

$$\begin{aligned} &D_1^{n-1}\{2\}\left(D_1\{2\}\mathsf{d}_3^- f_2 + n(D_2\{2\}\mathsf{d}_3^- B - \mathsf{d}_3^- K)\right) w_{\varkappa} \equiv \\ &\quad \equiv D_1^{n-1}\{2\}\left(D_1\{2\}e_0^- w_{(n,0,0)} + D_2\{2\}e_0^- w_{(n-1,1,0)}\right) \quad \bmod \mathbf{dlex}(< {\scriptstyle (n-1,1,0)})\,. \end{aligned}$$

It is not difficult to prove that this implies the following congruence

$$\begin{aligned} &(D_1\{2\}\mathsf{d}_3^- f_2 + n(D_2\{2\}\mathsf{d}_3^- B - \mathsf{d}_3^- K)\,)w_{\varkappa} \equiv \\ &\quad \equiv D_1\{2\}e_0^- w_{(n,0,0)} + D_2\{2\}e_0^- w_{(n-1,1,0)} \quad \bmod \mathbf{dlex}(\le {\scriptstyle (0,0,1)})\,. \end{aligned}$$

This means that the coefficients by $\widehat{\partial}_1$ and $\widehat{\partial}_2$ in the above expression on the left and on the right are equal. Clearly $\mathsf{d}_3^- K\, w_{\varkappa} = (p+2)\widehat{\partial}_1 \mathsf{d}_3^- f_2 w_{\varkappa} - \widehat{\partial}_2 \mathsf{d}_3^- (f_{12} - f_1 f_2) w_{\varkappa}$. Thus the equality of coefficients by $\widehat{\partial}_1$ gives

$$h_1(h'+2)\mathsf{d}_3^- f_2 w_{\varkappa} - n(p+2)\mathsf{d}_3^- f_2 w_{\varkappa} = h_1(h'+2) e_0^- w_{(n,0,0)}\,.$$

As $\mathrm{wt}_3\, \mathsf{d}_3^- f_2 w_{\varkappa} = (p+1,-2)$ we come to the equation

$$(p+1-n)\,\mathsf{d}_3^- f_2 w_{\varkappa} = (p+1)\, e_0^- w_{(n,0,0)}\,. \tag{4.8}$$

For $\widehat{\partial}_2$ taking into account weights we get

$$\left(f_1(p+2)\mathsf{d}_3^- f_2 + n\mathsf{d}_3^- B + n\mathsf{d}_3^- (f_{12} - f_1 f_2)\right) w_{\varkappa} = f_1(p+2) e_0^- w_{(n,0,0)} + e_0^- w_{(n-1,1,0)}$$

Or $\quad \mathsf{d}_3^- (n\,B + n(f_{12} - f_1 f_2) + (p+2) f_1 f_2) w_{\varkappa} = e_0^- w_{(n-1,1,0)} + (p+2) f_1 e_0^- w_{(n,0,0)}$, which is the same as

$$(p+2)\,\mathsf{d}_3^-\,(n f_{12} + f_1 f_2)\, w_{\varkappa} = e_0^- w_{(n-1,1,0)} + (p+2) f_1 e_0^- w_{(n,0,0)}. \tag{4.9}$$

Notice that

$$\mathsf{d}_3^- f_2 w_{\varkappa} = \mathsf{d}_{31}^- f_{12}(\mathsf{d}_1^+ x_1^p) \qquad \text{and} \qquad \mathsf{d}_3^- f_{12} w_{\varkappa} = (\mathsf{d}_{31}^- f_1 - (p+1)\mathsf{d}_{32}^-) f_{12}(\mathsf{d}_1^+ x_1^p)\,, \tag{4.10}$$

therefore from (4.8) it follows that

$$e_0^- w_{(n,0,0)} = \frac{p+1-n}{p+1}\,\mathsf{d}_{31}^- f_{12}(\mathsf{d}_1^+ x_1^p)\,. \tag{4.11}$$

On the other hand it is clear that $\mathrm{wt}_3\, w_{(n,0,0)} = (p+1,0)$ and $w_{(n,0,0)} \in \Lambda_2^-\Lambda_2^+ V$, hence by Proposition 3.13 we get $w_{(n,0,0)} = c_1\, \mathsf{d}_1^- \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^+ \mathsf{d}_\bullet^+) x_1^p$. Then a straightforward calculation shows that

$$e_0^- w_{(n,0,0)} = c_1\, \mathsf{d}_{13}^- f_{12}(\mathsf{d}_1^+ x_1^p) = -c_1\, \mathsf{d}_{31}^- f_{12}(\mathsf{d}_1^+ x_1^p)\,.$$

Comparing with (4.11) we get $c_1 = -\frac{p+1-n}{p+1}$.

Similarly from (4.9) we get

(4.12)
$$e_0^- w_{(n-1,1,0)} = (p+2)\left(\mathsf{d}_{31}^-(n+1+c_1) f_1 + \mathsf{d}_{32}^-(1-n(p+1)+c_1)\right) f_{12}(\mathsf{d}_1^+ x_1^p)\,.$$

Also from Proposition 3.13 we get

$$w_{\varkappa-(1)} = w_{(n-1,1,0)} = c_2\, \mathsf{d}_{12}^- \mathsf{C}(\mathsf{d}_\bullet^+ \mathsf{d}_\bullet^+ x_\bullet) x_1^{p-1} + c_3\, \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_{12}^+\, x_\bullet) x_1^{p-1}\,. \tag{4.13}$$

Hence $e_0^- w_{(n-1,1,0)}$ can be written as follows.

$$\begin{aligned} e_0^- w_{(n-1,1,0)} = \quad & c_2\, \mathsf{d}_{12}^-((p-1)\mathsf{d}_1^+ x_3^2 x_1^{p-2} - p\, \mathsf{d}_3^+ x_3 x_1^{p-1}) \\ + \quad & c_3\, \mathsf{d}_{12}^-((p-1)\mathsf{d}_1^+ x_3^2 x_1^{p-2} + \mathsf{d}_3^+ x_3 x_1^{p-1}) \\ + \quad & c_3\, \mathsf{d}_{23}^-(\qquad p\, \mathsf{d}_1^+ x_3 x_1^{p-1} + \mathsf{d}_3^+ x_1^p) \\ + \quad & c_3\, \mathsf{d}_{31}^-((p-1)\mathsf{d}_1^+ x_2 x_3 x_1^{p-2} + \mathsf{d}_2^+ x_3 x_1^{p-1} + \mathsf{d}_3^+ x_2 x_1^{p-1})\,. \end{aligned} \tag{4.14}$$

From this and (4.12) we conclude that

$$\begin{aligned} (p-1)(c_2+c_3) &= 0\,, \\ -p\, c_2 + c_3 &= 0\,. \end{aligned}$$

We see that either $w_{(n-1,1,0)} = 0$ or $p=1$, $n=1$, because $n = \varkappa_1 \le p$.

Let us consider the first alternative. Here (4.12) reduces to the equation

$$(n+1+c_1)\, \mathsf{d}_{31}^- f_1 f_{12}(\mathsf{d}_1^+ x_1^p) \quad = \quad (n(p+1)-1-c_1)\, \mathsf{d}_{32}^- f_{12}(\mathsf{d}_1^+ x_1^p)\,,$$

which implies $n+1+c_1 = n(p+1)-1-c_1 = 0$, therefore $n(p+2)=0$, hence $n=0$ and $\varkappa = (0,1,0)$. This clearly gives us Theorem 3.15(4).

For the second alternative $c_1 = -\frac{1}{2}$, and we get $c_2 = c_3 = \frac{9}{2}$ matching (4.12) with (4.14). Let us observe that $\mathrm{wt}_3\, w = \mathrm{wt}_3\, w_\varkappa + (0,-1) = (1,0)$ and

$$w = D_1 D_2 w_\varkappa + D_1 w_{(1,0,0)} + D_2 w_{(0,1,0)} + w_{(0,0,0)}\,,$$

with no other terms possible. For $w_{(0,0,0)}$ we have $w_{(0,0,0)} \in \Lambda_3^-\Lambda_3^+ V$, hence $w_{(0,0,0)} = c_4 \mathsf{d}_{123}^- \mathsf{d}_{123}^+ x_1$. Therefore we have the information to compute $e_0^- w$ quite explicitly. It takes some effort to check that it is not equal to zero whatever c_4. We get no quasi-singular vectors.

Subcase: $\varkappa_2 = 0$.
Now $\varkappa = (N,0,0)$, therefore

$$N\, D_1^{N-1}\{2\}\mathsf{d}_3^- w_\varkappa \quad \equiv \quad \sum_{|\alpha|=N-1} D\{2\}^\alpha e_0^- w_\alpha \bmod \deg_{\mathfrak{s}}(< N-1)\,.$$

The expression (4.13) still holds for $w_{\varkappa-(1)}$, and similarly we cannot have d_{12}^- in $e_0^- w_{\varkappa-(1)}$, therefore we are bound to conclude that $p=1$, $N=1$. This leads us to

the quasi-singular vector from Theorem 3.15(7).

Case 2: $\underline{w_\varkappa = \mathsf{d}_1^- \mathsf{d}_1^+ x_1^p}$.
Here $\varkappa_1 \le p+2$, $\varkappa_2 = 0$ by Remark 4.3.

Lemma 4.8. $\varkappa_3 = 0$. *For any* $\varkappa_1 \le p+2$ *the vector exists.*

Proof. It is clear from (4.3) that if $\varkappa_3 \neq 0$, then $e_0^- w_{\varkappa-(3)} \neq 0$ and in this case $\mathrm{wt}_3\, w_{\varkappa-(3)} = (p+2,1,0)$. A vector with such weight is not present in the list of Proposition 3.17. Hence $\varkappa_3 = 0$. Now the expression given in Theorem 3.15(1) shows that for any $\varkappa = (\varkappa_1, 0, 0)$ a vector w exists. □

Case 3: $\underline{w_\varkappa = \mathsf{d}_{123}^+,\ \mathsf{d}_1^- \mathsf{d}_{123}^+,\ \mathsf{d}_{12}^- \mathsf{d}_{123}^+,\ \mathsf{d}_{123}^- \mathsf{d}_{123}^+}$.
Form (4.1) we conclude that $\deg_\Lambda w_\alpha \ge \deg_\Lambda w_\varkappa + 2 \ge 5$ for any $\alpha \neq \varkappa$, and that $\mathrm{wt}_2\, w_\alpha = \mathrm{wt}_2\, w_\varkappa$. One immediately checks that this leaves no options but $w_\alpha = 0$. Therefore $w = D^\varkappa w_\varkappa$.

For $w_\varkappa = \mathsf{d}_{123}^- \mathsf{d}_{123}^+$, it follows from Remark 4.3 that $\varkappa_1 = \varkappa_2 = 0$. After that $\varkappa_3$ is arbitrary as the vectors given by Theorem 3.15(2) show.

For the rest of the vectors $\mathsf{d}_3^- w_\varkappa \neq 0$. Now, because $w = w_{\mathrm{top}}$, from (4.3) it follows that $\varkappa_3 = 0$. Here $\varkappa_1, \varkappa_2 \le 1$ by Remark 4.3, and we are left with pretty limited choices. It is easy to check that all what we get are vectors listed in Theorem 3.15(9) and (10).

We have gone through all the cases of Proposition 4.2, thus the proof of Theorem 3.15 is accomplished.

5. **Proof of Theorem 3.16.**

We shall work with vectors given by Theorem 3.15. Let us tabulate the weights of the vectors belonging to various cases of the theorem:

case	p	wt_3	wt_2	wt_1
(1)	$p \ge 0$	$(2+p-n, 0)$	0	$-\frac{2(n+1)}{3}$
(2)	$p = 0$	$(0, n)$	0	$-\frac{2(n+3)}{3}$
(3)	$p = 1$	$(0,0)$	1	$-\frac{5}{3}$
(4)	$p \ge 0$	$(p+1, 0)$	0	$-\frac{4}{3}$
(5)	$p = 1$	$(0,0)$	0	$-\frac{2}{3}$
(6)	$p = 1$	$(1,0)$	1	-1
(7)	$p = 1$	$(0,1)$	0	$-\frac{4}{3}$
(8)	$p = 0$	$(0,0)$	1	-1
(9)	$p = 0$	$(0,0)$	2	-2
(10)	$p = 0$	$(1,0)$	1	$-\frac{7}{3}$

There are coincidences in the table, namely (1) for $n = 1$ gives the same values as (4) and that is all. We are to check how e_0^+ acts on a quasi-singular vector which has to be a weight vector and a linear combination of the vectors given in the various cases of Theorem 3.15. We conclude that the only linear combination to consider is (1) for $n = 1$ and (4), the other cases are to be considered separately. We will go case by case.

Case (1).
Here we consider the semi-singular vector of the form

$$\begin{aligned} &(5.1)\quad w_{(1)} = D_1^n u_1 - a\, D_1^{n-1} u_2 - b\, D_1^{n-2} u_3\,, \quad \text{where} \\ &u_1 = \mathsf{d}_1^- \mathsf{d}_1^+ \, x_1^p\,, \qquad u_2 = \mathsf{d}_1^- (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{23} \mathsf{d}_1^+ \, x_1^p\,, \qquad u_3 = \mathsf{d}_{123}^- \mathsf{d}_{123}^+ \, x_1^p\,. \end{aligned}$$

and $a = n(p+3)$, $b = n^{[2]}(p+3)^{[2]}$. By Proposition 3.7 we can write

$$\begin{aligned} (5.2)\ e_0^+ \, w_{(1)} \;&=\; D_1\{2\}^n e_0^+ u_1 - aD_1\{2\}^{n-1}(e_0^+ u_2 + \tfrac{1}{p+3} \mathsf{d}_3^+ B\, u_1) \\ &\quad -bD_1\{2\}^{n-2}(e_0^+ u_3 - \tfrac{1}{p+2} \mathsf{d}_3^+ B\, u_2) + b(n-2) D_1\{2\}^{n-3}(\mathsf{d}_3^+ B\, u_3)\,. \end{aligned}$$

Let us compute the terms.

Lemma 5.1. *a.* $e_0^+ u_1 = 0$,
b. $e_0^+ u_2 + \frac{1}{p+3} \mathsf{d}_3^+ B\, u_1 \equiv -\widehat{\partial}_2 f_{12}(\mathsf{d}_1^+ \, x_1^p) \mod \deg_{\mathsf{S}}(< 1)$,
c. $e_0^+ u_3 - \frac{1}{p+2} \mathsf{d}_3^+ B\, u_2 \equiv 0 \mod \deg_{\mathsf{S}}(\le 1)$,
d. $\mathsf{d}_3^+ B\, u_3 \equiv 0 \mod \deg_{\mathsf{S}}(\le 1)$.

Proof. For a. we have

$$e_0^+ \, u_1 = e_0^+ \mathsf{d}_1^- \mathsf{d}_1^+ \, x_1^p = -f_2 \, \mathsf{d}_1^+ \, x_1^p + 0 = 0\,.$$

Now

$$e_0^+ \, u_2 = e_0^+ \mathsf{d}_1^- (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{23} \mathsf{d}_1^+ \, x_1^p = -f_2 (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{23} \mathsf{d}_1^+ \, x_1 - \mathsf{d}_1^- \, f_{12} \mathsf{d}_{31}^+ \, x_1^p = -p\, \mathsf{d}_1^- \mathsf{d}_{31}^+ \, x_3 x_1^{p-1}\,,$$

and

$$\begin{aligned} \tfrac{1}{p+3} \mathsf{d}_3^+ B\, u_1 \;&=\; \mathsf{d}_3^+ f_{12}(\mathsf{d}_1^- \mathsf{d}_1^+ \, x_1^p) \\ &=\; \mathsf{d}_3^+ \mathsf{d}_1^- f_{12}(\mathsf{d}_1^+ \, x_1^p) + \mathsf{d}_3^+ \mathsf{d}_3^- \mathsf{d}_1^+ \, x_1^p \equiv -\widehat{\partial}_2 f_{12}(\mathsf{d}_1^+ \, x_1^p) \mod \deg_{\mathsf{S}}(< 1)\,, \end{aligned}$$

hence b. follows. In calculation with c. and d. we shall similarly move d_3^+ over d_i^- that can make terms with no more than one $\widehat{\partial}_k$, thus the statements follow. $\square$

It follows from the lemma and (5.2) that

$$(5.3)\qquad e_0^+ \, w_{(1)} \equiv aD_1\{2\}^{n-1} \widehat{\partial}_2 f_{12}(\mathsf{d}_1^+ \, x_1^p) \mod \deg_{\mathsf{S}}(< n)\,.$$

We can apply Proposition 3.5 with $\alpha=(n-1,0,0)$, $s=2$, thus $b=k=0$ in the summation on the right hand side. Let us notice that

$$\begin{aligned}
&f_1^iB^j(h_1-i-j)^{[n-1-i-j]}(h'\{2\}-j)^{[n-1-j]}\widehat{\partial}_2f_{12}(\mathsf{d}_1^+\,x_1^p)\\
&\quad= (p+1-i-j)^{[n-1-i-j]}(p+2-j)^{[n-1-j]}f_1^iB^j\widehat{\partial}_2f_{12}(\mathsf{d}_1^+\,x_1^p)\\
&\quad= (p+1-i-j)^{[n-1-i-j]}(p+2-j)^{[n-1-j]}\,(p+2)^{[j]}\,f_1^i\widehat{\partial}_2f_{12}^{j+1}(\mathsf{d}_1^+\,x_1^p)\\
&\quad= (p+1-i-j)^{[n-1-i-j]}(p+2)^{[n-1]}(-i\widehat{\partial}_1f_1^{i-1}+\widehat{\partial}_2f_1^i)f_{12}^{j+1}(\mathsf{d}_1^+\,x_1^p)\,.
\end{aligned}$$

We see that in calculating "mod $\mathbf{dlex}(\le(n-2,1,1))$" we need only to consider the terms with $i,j=0,0$, $i,j=1,0$, $i,j=2,0$, and $i,j=1,1$, therefore "mod $\mathbf{dlex}(\le(n-2,1,1))$"

$$\begin{aligned}
e_0^+\,w_{(1)}\equiv C\quad&\left(\binom{n-1}{0}(p+1)^{[n-1]}\widehat{\partial}_1^{\,n-1}(\,\widehat{\partial}_2)\right.\\
&+\binom{n-1}{1}\,p^{[n-2]}\;\widehat{\partial}_1^{\,n-2}\widehat{\partial}_2(-\widehat{\partial}_1+\widehat{\partial}_2f_1)\\
&+\binom{n-1}{2}(p-1)^{[n-3]}\widehat{\partial}_1^{\,n-3}\widehat{\partial}_2^{\,2}(-2\widehat{\partial}_1f_1+\widehat{\partial}_2f_1^2)\\
&\left.+\binom{n-1}{1,1}(p-1)^{[n-3]}\widehat{\partial}_1^{\,n-3}\widehat{\partial}_2\widehat{\partial}_3(-\widehat{\partial}_1+\widehat{\partial}_2f_1)f_{12}\right)f_{12}(\mathsf{d}_1^+\,x_1^p)\,,
\end{aligned}\tag{5.4}$$

where $C=a(p+2)^{[n-1]}\neq0$ because $0<n\le p+2$. We should keep in mind that some terms are absent for $n\le2$, namely for $n=1$ all terms but the first disappear, and for $n=2$ we have just the first two terms left.

Clearly for $n\ge3$, thus $p\ge n-2\ge1$, we can rewrite (5.4) as follows

$$\begin{aligned}
e_0^+\,w_{(1)}\equiv A\quad&\Big(p^{[n-2]}((p+1)-(n-1))\widehat{\partial}_1^{\,n-1}\widehat{\partial}_2\\
&+(p-1)^{[n-3]}(n-1)(p-(n-2))\widehat{\partial}_1^{\,n-2}\widehat{\partial}_2^{\,2}f_1\\
&-(n-1)(n-2)(p-1)^{[n-3]}\widehat{\partial}_1^{\,n-2}\widehat{\partial}_2\widehat{\partial}_3f_{12}\Big)\,f_{12}(\mathsf{d}_1^+\,x_1^p)\,.
\end{aligned}\tag{5.5}$$

Here $p^{[n-2]}\neq0$, $(p-1)^{[n-3]}\neq0$, and we notice that $f_{12}^2(\mathsf{d}_1^+\,x_1^p)\neq0$ because $p\ge1$. We conclude that $e_0^+\,w=0$ is not possible when $n\ge3$.

It is easy to see from (5.4) that $n=1$ is impossible too. We are left with $p=0$, $n=2$, which is in fact the case (ii) of Theorem 3.16, we should only notice that the vector $\mathsf{d}_{123}^-\mathsf{d}_{123}^+$ is itself semi-singular, it could be either added or not in the expression.

Cases (1)&(4).
We are to consider a linear combination $w=Aw_{(1)}+Bw_{(4)}$ where $w_{(1)}$ is given by

(5.1) with $n=1$ and $w_{(4)}$ is the vector of the case (4) of Theorem 3.15, thus

$$w_{(4)} = D_2\, v_1 - v_1\,, \text{ where} \tag{5.6}$$
$$v_1 = (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{12}\, x_1^p + p(\mathsf{d}_\circ^- \mathsf{d}_1^+ x_\circ)_{12}\, x_1^{p-1},$$
$$v_2 = \mathsf{d}_1^- (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{23}\, \mathsf{d}_1^+ x_1^p.$$

One can check that $e_0^+ v_1 = -(p+2) f_{12} \mathsf{d}_1^+\, x_1^p$, $e_0^+ v_2 = -p \mathsf{d}_1^- \mathsf{d}_{31}^+\, x_3 x_1^{p-1}$ and

$$\begin{aligned} e_0^+ w_{(4)} &= -(p+2)\widehat{\partial}_2 f_{12} \mathsf{d}_1^+\, x_1^p - \mathsf{d}_3^+ f_2 v_1 - e_0^+ v_2 \\ &= -(p+2)\widehat{\partial}_2 f_{12} \mathsf{d}_1^+\, x_1^p - \mathsf{d}_3^+ ((\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{13}\, x_1^p + p(\mathsf{d}_\circ^- \mathsf{d}_1^+ x_\circ)_{13}\, x_1^{p-1}) - e_0^+ v_2 \\ &= -(p+1)\widehat{\partial}_2 f_{12} \mathsf{d}_1^+\, x_1^p - (p+1)\mathsf{d}_3^- \mathsf{d}_{31}^+\, x_1^p + p\, \mathsf{d}_1^- \mathsf{d}_{31}^+\, x_3 x_1^{p-1}\,. \end{aligned} \tag{5.7}$$

On the other hand as we already calculated above

$$\begin{aligned} e_0^+ w_{(1)} &= -\mathsf{d}_3^+ B\, u_1 - (p+3) u_2 \\ &= (p+3)\widehat{\partial}_2 f_{12} \mathsf{d}_1^+\, x_1^p + (p+3)\mathsf{d}_3^- \mathsf{d}_{31}^+\, x_1^p + 2(p+3)p\, \mathsf{d}_1^- \mathsf{d}_{31}^+\, x_3 x_1^{p-1}\,. \end{aligned} \tag{5.8}$$

Therefore $e_0^+ w = 0$ implies that $A(p+3) = B(p+1)$ and $A\, 2(p+3)p = B\, p$. We conclude that $p=0$, $B=3A$, and the vector is proportional to the one of the case (iii) of Theorem 3.16.

In the remaining cases (2), (3) and (5)-(10) the calculations are straightforward, in (2), (3) and (5)-(7) we get no semi-singular vectors, the case (8) corresponds to (i), (9) to (iv), and (10) to (v).

6. **Proof of Theorem 3.19.**

As before, our considerations are based on (3.6)-(3.8), Proposition 3.11 and Remarks 3.12, 4.1, 4.3 We follow the same strategy that was used for finding quasi-singular vectors in Section 4, and we keep the notations and conventions made at the beginning of Section 4. But we are now in the context of Theorem 3.19, in particular $q \geq 1$.

Let us remind that by Proposition 3.11 the possible top level coefficients are listed in Corollary 3.18. Remark 4.1 helps us to describe them quite explicitely.

Proposition 6.1. (1): *If* $w_\varkappa = \partial_3^q,\ \overset{+}{\Delta}(\partial_*)\partial_3^{q-1}$, *or* $\overset{-}{\Delta}(\overset{+}{\Delta}(\partial_*)\partial_*)\partial_3^{q-2}$, $\quad q \geq 2$, *then* $I_{top} = \{\varkappa\}$.

(2): *If* $w_\varkappa = \mathsf{d}_1^- \overset{+}{\Delta}(\partial_*)\partial_3^{q-1}$ *then either* $I_{top} = \{\varkappa\}$ *or* $I_{top} = \{\varkappa, \sigma\}$, *where* $\sigma = \varkappa + (-1,0,1)$ *and* $w_\sigma = c_\sigma \overset{-}{\Delta}(\overset{+}{\Delta}(\partial_*)\partial_*)\partial_3^{q-2}$, $c_\sigma \in \mathbb{C}$, $\quad q \geq 2$.

(3): *If* $q=1$ *and* $w_\varkappa = \mathsf{d}_1^+ \overset{+}{\Delta}(\partial_*)$, $\quad (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{12} \overset{+}{\Delta}(\partial_*)$, $\quad \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+)\overset{+}{\Delta}(\partial_*)$, *then* $I_{top} = \{\varkappa\}$.

(4): *If* $q=1$ *and* $w_\varkappa = \mathsf{d}_{12}^- \mathsf{d}_1^+ \overset{+}{\Delta}(\partial_*)$, *then either* $I_{top} = \{\varkappa\}$ *or* $I_{top} = \{\varkappa, \sigma\}$, *where* $\sigma = \varkappa + (-1,0,1)$ *and* $w_\sigma = \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+)\overset{+}{\Delta}(\partial_*)$.

(5): *If* $q=1$ *and* $w_\varkappa = \mathsf{d}_1^- \mathsf{d}_1^+ \overset{+}{\Delta}(\partial_*)$, *then either* $I_{top} = \{\varkappa\}$ *or* $I_{top} = \{\varkappa, \sigma\}$, *where* $\sigma = \varkappa + (-1,1,0)$ *and* $w_\sigma = (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{12} \overset{+}{\Delta}(\partial_*)$.

Proof. What we need is to look at $s\ell(3)$-weights of vectors listed in Corollary 3.18 and use Remark 4.3. We leave the details to the reader. □

Lemma 6.2. *In the conditions of Proposition 6.1:*
1. Whatever I_{top} and $\varkappa$, one has

$$\sum_{|\alpha|=N-1} D\{2\}^{\alpha} e_0^- w_\alpha \equiv \varkappa_3\, D^{\varkappa-(3)}\{1\}\, \mathsf{d}_3^- w_{\varkappa-(3)} \quad \mod \mathbf{lex}(< \varkappa-(3))\,, \tag{6.1}$$

2. Whatever I_{top}, if $\varkappa_3 = 0$ then

$$\sum_{|\alpha|=N-1} D\{2\}^{\alpha} e_0^- w_\alpha \equiv \varkappa_2 \widehat{\partial}^{\varkappa-(2)} \mathsf{d}_3^- f_2((h_1+1)^{[\varkappa_1]} - \varkappa_1 h_1^{[\varkappa_1-1]}) h'^{[\varkappa_1]} (h_2-1)^{[\varkappa_2-1]} w_\varkappa \mod \mathbf{lex}(< \varkappa-(2)), \tag{6.2}$$

Proof. Let us notice that the general formula (4.2) is valid in our context. To compute the RHS we use Proposition 3.7 for $e_0^- D^\varkappa w_\varkappa$ and $e_0^- D^\sigma w_\sigma$.

But clearly $\sigma-(3) <_{\text{lex}} \varkappa-(3)$ so we get (1). For (2) we have to check that $\sigma-(3)$ and $\sigma-(2)$ are strictly less than $\varkappa-(2)$, which is clear. Then we use Corollaries 3.6, 3.10. □

Now we begin to consider various possibilities for $w_\varkappa$ **case by case** according to Proposition 6.1.

Case 1(i): $\underline{w_\varkappa = \partial_3^q}$.
By Remark 4.3 $\varkappa_1 = 0$.

Lemma 6.3. $\varkappa_3 = 0$.

Proof. From (6.1) it follows that either $\varkappa_3 = 0$ or $e_0^- w_{\varkappa-(3)} \neq 0$. Checking with the list of Proposition 3.17 with the weights in mind we see that the latter is impossible. □

We conclude that $\varkappa = (0, N, 0)$ and now (6.2) is valid with the RHS equal to

$$\widehat{\partial}^{\varkappa-(2)} \mathsf{d}_3^- f_2 (h_2-1)^{[\varkappa_2-1]} w_\varkappa = -q^{[N]}\, \widehat{\partial}^{\varkappa-(2)} \mathsf{d}_3^- \partial_2 \partial_3^{q-1}\,,$$

and $q^{[N]} \neq 0$ because $N \leq q$.

The weight of $w_{\varkappa-(2)}$ is easy to determine and Proposition 3.17 provides a vector of this weight

$$w_{\varkappa-(2)} = c_1 \mathsf{d}_1^- \overset{+}{\Delta}(\partial_*) \partial_3^{q-1} + c_2 \overset{-}{\Delta}(\mathsf{d}_1^+ \partial_*) \partial_3^{q-1}\,. \tag{6.3}$$

We can assume that $c_1 = 0$ because $e_0^- \mathsf{d}_1^- \overset{+}{\Delta}(\partial_*) \partial_3^{q-1} = 0$. Then $e_0^- w_{\varkappa-(2)} = c_2((q-1)\overset{-}{\Delta}(\partial_*)\partial_2\partial_3^{q-2} + \mathsf{d}_3^- \partial_2 \partial_3^{q-1})$. Substituting this into (6.2) we get $q = 1$, $N = 1$, and $c_2 = -1$. We have come to the vector from case (5) of Theorem 3.19.

Case 1(ii): $\underline{w_\varkappa = \overset{+}{\Delta}(\partial_*)\partial_3^{q-1}}$.
As $\text{wt}_3\, w_\varkappa = (0, q-1)$, Remark 4.3 implies that $\varkappa_1 = 0$. Let us prove that $\varkappa_3 = 0$.

We shall use (6.1). Let us consider what is possible for $w_{\varkappa-(3)}$. By Proposition 3.17 and with weights in mind we get

$$w_{\varkappa-(3)} = c_1(\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\overset{+}{\Delta}(\partial_*)\partial_3^{q-1} + c_2\mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)\partial_3^q\,. \tag{6.4}$$

Immediate calculation shows that

$$\begin{array}{rcl} e_0^-(\mathsf{d}_\circ^-\mathsf{d}_\circ^+)_{12}\overset{+}{\Delta}(\partial_*)\partial_3^{q-1} & = & -(q-1)(\mathsf{d}_1^-\overset{+}{\Delta}(\partial_*)\partial_1 + \mathsf{d}_2^-\overset{+}{\Delta}(\partial_*)\partial_2)\partial_3^{q-2}\,, \\ e_0^-\mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+\mathsf{d}_\bullet^+)\partial_3^q & = & -q\overset{-}{\Delta}(\mathsf{d}_3^+\partial_*)\partial_3^{q-1} + \mathsf{d}_3^-(\ldots)\,. \end{array} \tag{6.5}$$

It follows from (6.1) that the only possibility is $\varkappa_3 = c_1 = c_2 = 0$.

We conclude that $\varkappa = (0, N, 0)$. Here $1 \le N \le q-1$ by Remark 4.3, in particular $q \ge 2$. Again we can use (6.2), where the coefficient by $\widehat{\partial}^{\varkappa-(2)}$ in the RHS is

$$\mathsf{d}_3^- f_2(h_2-1)^{[\varkappa_2-1]}w_\varkappa = -(q-1)^{[N]}\mathsf{d}_3^-\overset{+}{\Delta}(\partial_*)\partial_2\partial_3^{q-1}\,, \tag{6.6}$$

and of course $(q-1)^{[N]} \neq 0$. On the other hand from Proposition 3.17 it follows that

$$w_{\varkappa-(2)} = c\overset{-}{\Delta}(\mathsf{d}_1^+\overset{+}{\Delta}(\partial_*)\partial_*)\partial_3^{q-2}\,, \tag{6.7}$$

and that implies

$$e_0^- w_{\varkappa-(2)} = -c(q-2)\overset{-}{\Delta}(\mathsf{d}_1^+\overset{+}{\Delta}(\partial_*)\partial_*)\partial_2\partial_3^{q-3} + \mathsf{d}_3^-(\ldots)\,. \tag{6.8}$$

Substituting results of (6.6) and (6.7) into (6.2) we arrive at a contradiction. Therefore no w exists in this case.

Case 1(iii): $\underline{w_\varkappa = \overset{-}{\Delta}(\overset{+}{\Delta}(\partial_*)\partial_*)\partial_3^{q-2}}$.

Now $\mathrm{wt}_3\, w_\varkappa = (0, q-2)$ and again Remark 4.3 implies that $\varkappa_1 = 0$ and that for $q = 2$ also $\varkappa_2 = 0$. Let us prove that $\varkappa_2 = 0$ for $q > 2$ as well. Then we come to the vector given by Theorem 3.19(1).

Denote $\varkappa = (0, s, t)$ and suppose $s \ge 1$. From Proposition 3.17 with our weigths in mind we get

$$w_{\varkappa-(3)} = c\,\mathsf{C}(\mathsf{d}_\bullet^-\mathsf{d}_\bullet^-\mathsf{d}_\bullet^+)\overset{+}{\Delta}(\partial_*)\partial_3^{q-1}\,, \tag{6.9}$$

and $w_{\varkappa-(2)} = 0$. By Proposition 3.5

$$D^\varkappa w_\varkappa = D_2^s D_3^t w_\varkappa = \sum_{k=0}^{s}\binom{s}{k}\widehat{\partial}_2^{s-k}\widehat{\partial}_3^{t+k}(q-2-k)^{[s-k]}f_2^k w_\varkappa\,,$$

therefore

$$e_0^- D^\varkappa w_\varkappa = \sum_{k=0}^{s}\binom{s}{k}(t+k)\widehat{\partial}_2^{s-k}\widehat{\partial}_3^{t+k-1}(q-2-k)^{[s-k]}(-\mathsf{d}_3^- f_2^k w_\varkappa)\,. \tag{6.10}$$

Similarly

$$\begin{aligned} e_0^- D^{\varkappa-(3)}w_{\varkappa-(3)} = {}& c\sum_{k=0}^{s}\binom{s}{k}\widehat{\partial}_2^{s-k}\widehat{\partial}_3^{t+k-1}(q-1-k)^{[s-k]}f_2^k e_0^- w_{\varkappa-(3)} \\ & +c\sum_{k=0}^{s}\binom{s}{k}(t+k-1)\widehat{\partial}_2^{s-k}\widehat{\partial}_3^{t+k-2}(q-1-k)^{[s-k]}(-\mathsf{d}_3^- f_2^k w_{\varkappa-(3)})\,. \end{aligned} \tag{6.11}$$

Because $\varkappa_1 = 0$ and $w_{\varkappa-(2)} = 0$ we have

$$e_0^- D^{\varkappa} w_{\varkappa} + e_0^- D^{\varkappa-(3)} w_{\varkappa-(3)} \equiv 0 \quad \mod \deg_{\mathsf{s}}(< N-1)\,. \tag{6.12}$$

Looking at coefficients by $\widehat{\partial}_2^{s-k}\widehat{\partial}_3^{t+k-1}$ for $k = 0, 1$ in (6.10), (6.11) we get from (6.12)

$$\begin{aligned} (q-1)^{[s]}\, e_0^- w_{\varkappa-(3)} &= t\,(q-2)^{[s]}\, \mathsf{d}_3^- w_{\varkappa}\,, \\ (q-2)^{[s-1]} f_2\, e_0^- w_{\varkappa-(3)} &= (t+1)(q-3)^{[s-1]} \mathsf{d}_3^- f_2 w_{\varkappa}\,. \end{aligned}$$

But $e_0^- w_{\varkappa-(3)} = (q-1)\mathsf{d}_3^- \Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}$, therefore we come to

$$\begin{aligned} c\,(q-1)^2 &= t\,(q-1-s)\,, \\ c\,(q-1)(q-2) &= (t+1)(q-1-s)\,. \end{aligned}$$

Clearly the system implies $(q-2)\,t = (q-1)(t+1)$ which has no positive solutions. Thus $\varkappa_2 > 0$ is impossible, as we claimed.

Case 2: $\underline{w_{\varkappa} = \mathsf{d}_1^- \Delta^+(\partial_*)\partial_3^{q-1}}$.
We have $\mathrm{wt}_3\, w_{\varkappa} = (1, q-1)$ and Remark 4.3 implies that $\varkappa_1 = 0$ or 1, and $\varkappa_2 \le q-1$.

Let us prove that $\varkappa_3 = 0$. Equation (6.1) is valid here with $\mathsf{d}_3^- w_{\varkappa} = \mathsf{d}_{31}^- \Delta^+(\partial_*)\partial_3^{q-1}$ in the RHS. Thus for $w_{\varkappa-(3)}$ we get

$$w_{\varkappa-(3)} = -c_1 \mathsf{d}_{12}^- \mathsf{d}_1^+ \Delta^+(\partial_*)\partial_3^{q-1} + c_2\, \mathsf{d}_1^- \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^+ \mathsf{d}_\bullet^+)\partial_3^{q}\,,$$

which coincides with the RHS of (6.4) multiplied by d_1^-. Therefore calculating $e_0^- w_{\varkappa-(3)}$ we can use the results in the RHS of (6.5) multiplied by d_1^-, and that gives

$$\begin{aligned} e_0^- w_{\varkappa-(3)} &= c_1(q-1)\,\mathsf{d}_{12}^- \Delta^+(\partial_*)\partial_2\partial_3^{q-2} - c_2\, q\, \mathsf{d}_1^- \Delta^-(\mathsf{d}_3^+ \partial_*)\partial_3^{q-1} &+ \mathsf{d}_{13}^-(\ldots) \\ &= c_1(q-1)\,\mathsf{d}_{12}^- \Delta^+(\partial_*)\partial_2\partial_3^{q-2} - c_2\, q\, \mathsf{d}_{12}^- \mathsf{d}_3^+ \partial_2\partial_3^{q-1} &+ \mathsf{d}_{13}^-(\ldots)\,. \end{aligned}$$

We conclude that $q = 1$, $c_2 = 0$, but then $e_0^- w_{\varkappa-(3)} = 0$ hence $\varkappa_3 = 0$.

We can now use (6.2) where in the RHS we have

$$\mathsf{d}_3^- f_2\, w_{\varkappa} = (q-1)\mathsf{d}_{13}^- \Delta^+(\partial_*)\partial_2\partial_3^{q-2}\,, \tag{6.13}$$

and for $w_{\varkappa-(2)}$ we get

$$w_{\varkappa-(2)} = c\,\mathsf{d}_1^- \Delta^-(\mathsf{d}_1^+ \Delta^+(\partial_*)\partial_*)\partial_3^{q-2}\,, \tag{6.14}$$

which of course is equal to the RHS of (6.7) multiplied by d_1^-.

Lemma 6.4. *If $q > 2$ then $\varkappa_2 = 0$.*

Proof. To calculate $e_0^- w_{\varkappa-(2)}$ we can use (6.8), and we get

$$e_0^- w_{\varkappa-(2)} = -\,c\,(q-2)\mathsf{d}_1^- \Delta^-(\mathsf{d}_1^+ \Delta^+(\partial_*)\partial_*)\partial_2\partial_3^{q-3} + \mathsf{d}_{13}^-(\ldots)\,. \tag{6.15}$$

Comparing it with (6.13) we see that if $q > 2$ then $c = \varkappa_2 = 0$. □

Notice that if $\varkappa_2 = 0$ then $\varkappa_1 = 1$ because $|\varkappa| \geq 0$, and for $\varkappa = (1,0,0)$, $q \geq 1$ the vector is given by Theorem 3.19(2). Because of Lemma 6.4, we are left to consider the situation where $q = 2$, $\varkappa_2 = 1$, $\varkappa_1 \leq 1$. Here for $\varkappa = (0,1,0)$ the vector is given by Theorem 3.19(4) and for $\varkappa = (1,1,0)$ by Theorem 3.19(3).

Case 3(i): $\underline{w_\varkappa = \mathsf{d}_1^+ \Delta^+(\partial_*)}$.
Here $\varkappa_2 = 0$, $\varkappa_1 \leq 1$ by Remark 4.3. From Proposition 3.17 we get $w_{\varkappa-(3)} = c\,\mathsf{d}_1^- \mathsf{d}_{123}^+ \partial_3$, then (6.1) shows that $c = \varkappa_3 = 0$.

We are bound to have $\varkappa = (1,0,0)$, and then the vector is given by Theorem 3.19(6).

Case 3(ii): $\underline{w_\varkappa = (\mathsf{d}_\circ^- \mathsf{d}_\circ^+)_{12} \Delta^+(\partial_*)}$.
We have $\mathrm{wt}_3\, w_\varkappa = (0,1)$, thus $\varkappa_1 = 0$, $\varkappa_2 \leq 1$ by Remark 4.3. Clearly $\mathrm{wt}_3\, w_{\varkappa-(3)} = (0,2)$, $\deg_\Lambda w_{\varkappa-(3)} = 5$, $\mathrm{wt}_2\, w_{\varkappa-(3)} = 1$, therefore from Proposition 3.17 we get $w_{\varkappa-(3)} = c\,\mathsf{d}_{12}^- \mathsf{d}_{123}^+ \partial_3$. Now (6.1) shows that $c = \varkappa_3 = 0$. We come to Theorem 3.19(7).

Case 3(iii): $\underline{w_\varkappa = \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+) \Delta^+(\partial_*)}$.
Clearly $\varkappa_1 = 0$, $\varkappa_2 = 0$ here, and we come to vectors given by Theorem 3.19(8).

Case 4: $\underline{w_\varkappa = \mathsf{d}_{12}^- \mathsf{d}_1^+ \Delta^+(\partial_*)}$.
Here $\varkappa_1 \leq 1$, $\varkappa_2 \leq 1$. From Proposition 3.17 we get $w_{\varkappa-(3)} = 0$ because there is no vector with the appropiate weight and $\deg_\Lambda w_{\varkappa-(3)} = 6$, hence $\varkappa_3 = 0$. Similarly $w_{\varkappa-(2)} = 0$ and $\varkappa_2 = 0$. Then $\varkappa = (1,0,0)$ and we have the vector in Theorem 3.19(9).

Case 5: $\underline{w_\varkappa = \mathsf{d}_1^- \mathsf{d}_1^+ \Delta^+(\partial_*)}$.
We see that $\varkappa_2 = 0$, $\varkappa_1 \leq 2$ and $\varkappa_3 = 0$ because no $w_{\varkappa-(3)}$ could be found. We have come to Theorem 3.19(10).

This finishes the proof of Theorem 3.19.

7. **Proof of Theorem 3.20.**

We shall denote $w_{(i)}$ the vector in the form given by the case (i) of Theorem 3.19. Considering the weights of these vectors we see that equal weights are relatively rare, therefore few linear combinations are to be considered. Namely these linear combinations are:

(a) $w_{(1)}|_{n=1}$ and $w_{(2)}$,
(b) $w_{(2)}|_{q=1}$ and $w_{(8)}|_{n=0}$,
(c) $w_{(7)}$ and $w_{(10)}|_{n=1}$,
(d) $w_{(8)}|_{n=1}$ and $w_{(9)}$.

According to the definition we shall consider the action of e_0^+ on $w_{(i)}$ going through cases $i=1,\dots,10$ Theorem 3.19 with the special regard to possible linear combinations.

Case (1): According to the formula of Proposition 3.7

$$
\begin{aligned}
e_0^+ w_{(1)} \quad = \quad & D_3^n\, e_0^+ \Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2} - \\
-n\, D_3^{n-1} & \left(\mathsf{d}_3^+ \Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2} - \tfrac{1}{q-1} e_0^+ \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+)\Delta^+(\partial_*)\partial_3^{q-1}\right) + \\
-n^{[2]} D_3^{n-2} & \left(\tfrac{1}{q-1}\mathsf{d}_3^+ \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+)\Delta^+(\partial_*)\partial_3^{q-1} + \tfrac{1}{q^{[2]}} e_0^+ \mathsf{d}_{123}^- \mathsf{d}_{123}^+ \partial_3^q\right) - \\
-n^{[3]} D_3^{n-3} & \left(\tfrac{1}{q^{[2]}}\mathsf{d}_3^+ \mathsf{d}_{123}^- \mathsf{d}_{123}^+ \partial_3^q\right).
\end{aligned}
\tag{7.1}
$$

The term in the last line is clearly zero. Also $e_0^+\Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}=0$. Now along with calculating the action of e_0^+, we shall move d_3^+ over products of d_i^- at some places. Namely

$$
\begin{aligned}
&\mathsf{d}_3^+ \Delta^-(\Delta^+(\partial_*)\partial_*)\partial_3^{q-2} = \\
&\qquad (\widehat{\partial}_1\Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2\Delta^+(\partial_*)\partial_1)\,\partial_3^{q-2} - \Delta^-(\mathsf{d}_3^+\Delta^+(\partial_*)\partial_*)\partial_3^{q-2},
\end{aligned}
\tag{7.2}
$$

$$
\begin{aligned}
&e_0^+ \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+)\Delta^+(\partial_*)\partial_3^{q-1} = \\
&\qquad \mathsf{d}_3^+\mathsf{d}_3^+\Delta^+(\partial_*)\partial_3^{q-1} + (q-1)\Delta^-(\mathsf{d}_3^+\Delta^+(\partial_*)\partial_*)\partial_3^{q-2}.
\end{aligned}
\tag{7.3}
$$

Looking at the other terms in (7.1) we conclude that

$$
e_0^+ w_{(1)} = -n\,\widehat{\partial}_3^{\,n-1}(\widehat{\partial}_1\Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2\Delta^+(\partial_*)\partial_1)\partial_3^{q-2} + (\dots),
$$

there $(\dots)$ marks the terms with $\deg_{\mathsf{S}}$ less than n. This gives $e_0^+ w_{(1)} \neq 0$ for all possible $n\geq 0$, $q\geq 2$.

Case (2): Similarly from Proposition 3.7 it follows

$$
e_0^+ w_{(2)} = D_1\{2\}\, e_0^+ \mathsf{d}_1^-\Delta^+(\partial_*)\partial_3^{q-1} - \mathsf{d}_3^+ B\, \mathsf{d}_1^-\Delta^+(\partial_*)\partial_3^{q-1} + e_0^+ \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+)\Delta^+(\partial_*)\partial_3^{q-1}.
\tag{7.4}
$$

We are to calculate the terms. For the first two terms we get

$$
\begin{aligned}
D_1\{2\}\, e_0^+ \mathsf{d}_1^-\Delta^+(\partial_*)\partial_3^{q-1} &= D_1\{2\}\,(q-1)\Delta^+(\partial_*)\partial_2\partial_3^{q-2} = \\
&\qquad (q-1)(q+1)(\widehat{\partial}_1\Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2\Delta^+(\partial_*)\partial_1)\,\partial_3^{q-2},
\end{aligned}
\tag{7.5}
$$

$$
\begin{aligned}
-\mathsf{d}_3^+ B\, \mathsf{d}_1^-\Delta^+(\partial_*)\partial_3^{q-1} &= (q-1)(\widehat{\partial}_1\Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2\Delta^+(\partial_*)\partial_1)\,\partial_3^{q-2} \\
&\quad + (q+1)\mathsf{d}_3^+\mathsf{d}_3^+\Delta^+(\partial_*)\partial_3^{q-1} - (q-1)\Delta^-(\mathsf{d}_3^+\Delta^+(\partial_*)\partial_*)\partial_3^{q-2},
\end{aligned}
\tag{7.6}
$$

and the last term has been calculated in (7.3).

From (7.1)-(7.6) we conclude that no linear combinations of $w_{(2)}$ and $w_{(1)}|_{n=1}$ makes a semi-singular vector, but that

$$w = w_{(2)}|_{q=1} - 3\mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^-_\bullet \mathsf{d}^+_\bullet)\Delta^+(\partial_*) = D_1 \mathsf{d}^-_1 \Delta^+(\partial_*) - 2\mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^-_\bullet \mathsf{d}^+_\bullet)\Delta^+(\partial_*)$$

is indeed semi-singular. This vector is in fact a linear combination of $w_{(2)}|_{q=1}$ and $w_{(8)}|_{n=0}$, and it is the vector presented in Theorem 3.20(iii).

Case (3): In order to evaluate $e_0^+ w_{(3)}$ let us calculate $e_0^+ D_1 D_2\, \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3$ first. According to Proposition 3.7 we get the following four terms:

$$\begin{aligned} e_0^+ D_1 D_2\, \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \quad &= \quad D_1\{2\} D_2\{2\}\, e_0^+ \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 - D_2\{2\}\mathsf{d}^+_3 B \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \\ &\quad\; - D_1\{2\}\mathsf{d}^+_3 f_2 \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 + \mathsf{d}^+_3 K \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \,. \end{aligned}$$

Now it is practical to compute these terms mod $\deg_{\mathsf{S}}(<2)$.

$$\begin{aligned} D_1\{2\} D_2\{2\}\, e_0^+ \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \quad &\equiv \quad 3\,\widehat{\partial}_2(\widehat{\partial}_1 \Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2 \Delta^+(\partial_*)\partial_1)\,, \\ D_2\{2\}\mathsf{d}^+_3 B \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \quad &\equiv \quad \widehat{\partial}_2(\widehat{\partial}_1 \Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2 \Delta^+(\partial_*)\partial_1)\,, \\ D_1\{2\}\mathsf{d}^+_3 f_2 \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \quad &\equiv \quad -3\,\widehat{\partial}_2(\widehat{\partial}_1 \Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2 \Delta^+(\partial_*)\partial_1)\,, \\ \mathsf{d}^+_3 K \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \quad &\equiv \quad 2\,\widehat{\partial}_2(\widehat{\partial}_1 \Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2 \Delta^+(\partial_*)\partial_1)\,. \end{aligned}$$

Looking at the whole expression for $w_{(3)}$ we immediately conclude that

$$e_0^+ w_{(3)} \equiv e_0^+ D_1 D_2\, \mathsf{d}^-_1 \Delta^+(\partial_*)\partial_3 \equiv 3\,\widehat{\partial}_2(\widehat{\partial}_1 \Delta^+(\partial_*)\partial_2 - \widehat{\partial}_2 \Delta^+(\partial_*)\partial_1) \quad \mod \deg_{\mathsf{S}}(<2)\,.$$

Thus $e_0^+ w_{(3)} \neq 0$, and we get no semi-singular vector in this case.

Cases (4)-(6): Each case here gives a semi-singular vector. They coincide with those of Theorem 3.20(ii), (i) and (iv).

Cases (7) and (10): It is straightforward to check that

$$e_0^+ w_{(7)} = -\widehat{\partial}_2 \mathsf{d}^+_3 \Delta^+(\partial_*) - \mathsf{d}^-_3 \mathsf{d}^+_{312}\partial_2 \,.$$

At the same time for $w'_{(10)} = w_{(10)}|_{n=1}$ we get

$$e_0^+ w'_{(10)} = 3\,\widehat{\partial}_2 \mathsf{d}^+_3 \Delta^+(\partial_*) + 3\,\mathsf{d}^-_3 \mathsf{d}^+_{312}\partial_2 \,.$$

Clearly a linear combination $w = \frac{1}{3} w_{(7)} + w'_{(10)}$ gives the semi-singular vector. Up to notations this is the vector from Theorem 3.20(v). And $w = w_{(10)}|_{n=2}$ is a semi-singular vector given in Theorem 3.20(vi).

Cases (8) and (9): Here

$$\begin{aligned} e_0^+ w_{(8)} \quad &= \quad D_3^n e_0^+ \mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^-_\bullet \mathsf{d}^+_\bullet)\Delta^+(\partial_*) - n\, D_3^{n-1}\mathsf{d}^+_3 \mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^-_\bullet \mathsf{d}^+_\bullet)\Delta^+(\partial_*) - n\, D_3^{n-1} e_0^+ \mathsf{d}^-_{123}\mathsf{d}^+_{123}\partial_3 \\ &\quad\; + n(n-1) D_3^{n-2}\mathsf{d}^+_3 \mathsf{d}^-_{123}\mathsf{d}^+_{123}\partial_3 \\ &\equiv \quad n\widehat{\partial}_3^{\,n-1}\left(\widehat{\partial}_1(\mathsf{d}^-_\circ \mathsf{d}^+_\circ)_{13} + \widehat{\partial}_2(\mathsf{d}^-_\circ \mathsf{d}^+_\circ)_{23} + \widehat{\partial}_3 \mathsf{d}^-_3 \mathsf{d}^+_3\right)\Delta^+(\partial_*) \quad \mod \deg_{\mathsf{S}}(<n)\,. \end{aligned}$$

This gives no semi-simgular vector. And for $w_{(9)}$ we get

$$\begin{aligned} e_0^+ w_{(9)} &= D_1\{2\}\, e_0^+ \mathsf{d}_{12}^- \mathsf{d}_1^+ \Delta^+(\partial_*) - \mathsf{d}_3^+ B \mathsf{d}_{12}^- \mathsf{d}_1^+ \Delta^+(\partial_*) - 2\, e_0^+ \mathsf{d}_{312}^- \mathsf{d}_{312}^+ \partial_3 \\ &\equiv -\Big(\widehat{\partial}_1(\mathsf{d}_1^- \mathsf{d}_3^+ + 2\mathsf{d}_3^- \mathsf{d}_1^+) + \widehat{\partial}_2(\mathsf{d}_2^- \mathsf{d}_3^+ + 2\mathsf{d}_3^- \mathsf{d}_2^+) + 4\,\widehat{\partial}_3\, \mathsf{d}_3^- \mathsf{d}_3^+\Big)\, \Delta^+(\partial_*) \ \bmod \deg_{\mathsf{S}}(<1). \end{aligned}$$

We see that no linear combination of $w_{(9)}$ and $w_{(8)}|_{n=1}$ could provide a semi-simgular vector.

□

8. **From semi-singular to singular vectors.**

The results obtained in Sections 3-7 and Remark 2.7 permit us to write explicitely not only admissible, but all semi-singular vectors, as it is done in the propositions below. Then we begin to work on determining the singular vectors following the directions outlined in Section 2.

Let us have in mind that the grading of L_- extends to the grading on $U(L_-)\otimes_{\mathbb{C}} V$, where we suppose grading on V being trivial. Hence for an element w in $U(L_-)\otimes_{\mathbb{C}} V$ it is natural to define a degree $\deg_{\mathrm{U}} w$ to be equal to the grading *with the opposite sign.*

Proposition 8.1. *Let V be an irreducible $s\ell(3)$-submodule in $\mathbb{C}[x_1,x_2,x_3]$ generated by x_1^p. Then semi-singular $s\ell(2)$-weight vectors w in $U(L_-)\otimes_{\mathbb{C}} V$ are the following (up to linear combinations).*

$$\begin{array}{llll}
\deg_{\mathrm{U}} w = 0: & x_1^p\,, & & \\
\deg_{\mathrm{U}} w = 1: & \mathsf{d}_1^+ x_1^p\,, & \mathsf{d}_1^- x_1^p\,, & \\
\deg_{\mathrm{U}} w = 2: & \mathsf{d}_1^- \mathsf{d}_1^+ x_1^p\,, & & \\
\deg_{\mathrm{U}} w = 3: & a = \mathsf{d}_{123}^+\,, & b = f_3 a = \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^+ \mathsf{d}_\bullet^+) - \widehat{\Delta}(\mathsf{d}_*^+)\,, & \\
 & & c = \frac{1}{2} f_3 b = \mathsf{C}(\mathsf{d}_\bullet^- \mathsf{d}_\bullet^- \mathsf{d}_\bullet^+) - \widehat{\Delta}(\mathsf{d}_*^-)\,, & d = \mathsf{d}_{123}^-\,, \\
\deg_{\mathrm{U}} w = 4: & \mathsf{d}_1^- a\,, & \mathsf{d}_1^- b\,, \quad \mathsf{d}_1^- c\,, & \\
\deg_{\mathrm{U}} w = 6: & \widehat{\Delta}(\mathsf{d}_*^-)\, a\,, & \widehat{\Delta}(\mathsf{d}_*^-)\, b \quad \widehat{\Delta}(\mathsf{d}_*^-)\, c\,, & \\
 & \qquad \textit{and} & d\,a - a\,d = 2\,d\,a + \widehat{\Delta}(\mathsf{d}_*^-)\, b\,, & \\
\deg_{\mathrm{U}} w = 7: & \mathsf{d}_1^- \widehat{\Delta}(\mathsf{d}_*^-) a\,, & \mathsf{d}_1^- \widehat{\Delta}(\mathsf{d}_*^-) b. &
\end{array}$$

Let us fix and keep for the following the above the notations a, b, c, d for the elements of $U(L_-)$ (these notations were also used in [4]).

Remark 8.2. *One should have in mind that a, b, c, d make a standard weight vector basic for a four-dimensional $s\ell(2)$-representation with the relations*

$$\begin{aligned} f_3\, a = b\,, \quad f_3\, b = 2c\,, \quad f_3\, c = 3d \\ e_3\, b = 3a\,, \quad e_3\, c = 2b\,, \quad e_3\, d = c\,, \end{aligned} \tag{8.1}$$

and that for any $i = 1, 2, 3$

$$\mathsf{d}_i^+ a = 0\,,\ \mathsf{d}_i^+ b = -\mathsf{d}_i^- a\,,\ \mathsf{d}_i^+ c = -\mathsf{d}_i^- b\,,\ \mathsf{d}_i^+ d = -\mathsf{d}_i^- c\,,\ 0 = \mathsf{d}_i^- d\,. \tag{8.2}$$

Clearly the same rules are true for the multiplication on the other side. Also

$$\begin{gathered} a\,\overset{+}{\Delta}(\partial_*) = 0\,, \\ a\,\overset{-}{\Delta}(\partial_*) = -b\,\overset{+}{\Delta}(\partial_*)\,,\quad b\,\overset{-}{\Delta}(\partial_*) = -c\,\overset{+}{\Delta}(\partial_*)\,,\quad c\,\overset{-}{\Delta}(\partial_*) = -d\,\overset{+}{\Delta}(\partial_*)\,, \\ d\,\overset{-}{\Delta}(\partial_*) = 0\,, \end{gathered} \tag{8.3}$$

and the same is true for $\widehat{\Delta}(\mathsf{d}_*^-)$, $\widehat{\Delta}(\mathsf{d}_*^+)$, *and for the multiplication from the other side as well.*

Proof. We are to combine the vectors of Theorem 3.16 with those from Corollary 3.14 that are semi-singular, apply φ according to Remark 2.7. In fact the notations of the proposition are written in a way is more convenient for the future calculations, namely the vectors are such that they are weight vectors and make the bases for irreducible $s\ell(2)$-representations. In particular from the relations

$$\begin{aligned} &a\,b = 0, && b\,a = 0, && a\,c = c\,a, && b\,c - c\,b = 3(a\,d - d\,a), \\ &b\,d = 0, && d\,b = 0, && b\,d = d\,b, && a\,d + d\,a = -\widehat{\Delta}(\mathsf{d}_*^-)\,b, \end{aligned} \tag{8.4}$$

it follows that $f_3(a\,d - d\,a) = e_3(a\,d - d\,a) = 0$ and thus the vector $d\,a - a\,d = 2\,d\,a + \widehat{\Delta}(\mathsf{d}_*^-)\,b$ makes an one-dimensional $s\ell(2)$-representation. We leave the rest for the reader. □

Similarly Theorem 3.20 and Corollary 3.21 lead us to the following proposition.

Proposition 8.3. *Let* V *be an irreducible* $s\ell(3)$*-submodule in* $\mathbb{C}[\partial_1, \partial_2, \partial_3]$ *generated by* ∂_3^q*. The semi-singular* $s\ell(2)$*-weight vectors* w *in* $U(L_-) \otimes_{\mathbb{C}} V$*, up to linear combinations, are:*

$$\begin{aligned}
&\deg_{\mathrm{U}} w = 0: && \partial_3^q\,, \\
&\deg_{\mathrm{U}} w = 1: && \overset{+}{\Delta}(\partial_*)\partial_3^{q-1}\,, \qquad \overset{-}{\Delta}(\partial_*)\partial_3^{q-1}\,, \\
&\deg_{\mathrm{U}} w = 2: && \mathsf{d}_1^+\overset{+}{\Delta}(\partial_*)\,, \qquad \mathsf{d}_1^-\overset{+}{\Delta}(\partial_*)\,,\quad \mathsf{d}_1^-\overset{-}{\Delta}(\partial_*)\,, \\
& && \quad \widehat{\partial}_2\partial_2 - \widehat{\partial}_3\partial_3 - \overset{-}{\Delta}(\mathsf{d}_1^+\partial_*) = \mathsf{d}_1^+\overset{-}{\Delta}(\partial_*), \qquad \textit{and} \quad \overset{-}{\Delta}(\overset{+}{\Delta}(\partial_*)\,\partial_*)\,, \\
&\deg_{\mathrm{U}} w = 3: && \mathsf{d}_1^-\mathsf{d}_1^+\overset{+}{\Delta}(\partial_*)\,, \\
&\deg_{\mathrm{U}} w = 4: && b\,\overset{+}{\Delta}(\partial_*)\,, \qquad c\,\overset{+}{\Delta}(\partial_*)\,, \qquad d\,\overset{+}{\Delta}(\partial_*)\,, \qquad \textit{and} \\
& && \mathsf{d}_1^-\mathsf{d}_1^+\overset{-}{\Delta}(\overset{+}{\Delta}(\partial_*)\partial_*) = \widehat{\partial}_2\mathsf{d}_1^-\overset{+}{\Delta}(\partial_*)\partial_2 - \widehat{\partial}_3\mathsf{d}_1^-\overset{+}{\Delta}(\partial_*)\partial_3 - \mathsf{d}_1^-\overset{-}{\Delta}(\mathsf{d}_1^+\overset{+}{\Delta}(\partial_*)\partial_*)\,, \\
&\deg_{\mathrm{U}} w = 5: && \mathsf{d}_1^-\,a\,\overset{-}{\Delta}(\partial_*)\,, \qquad \mathsf{d}_1^-\,b\,\overset{-}{\Delta}(\partial_*)\,, \\
&\deg_{\mathrm{U}} w = 7: && \widehat{\Delta}(\mathsf{d}_*^-)\,b\,\overset{+}{\Delta}(\partial_*)\,,\quad \widehat{\Delta}(\mathsf{d}_*^-)\,c\,\overset{+}{\Delta}(\partial_*)\,.
\end{aligned}$$

Proof. Let us notice that the vectors with $\deg_{\mathrm{U}} w \geq 4$ in the proposition are not all written in the $\Lambda^-\Lambda^+$ order. For the proof it is better to rewrite them in this order,

namely

$$
\begin{aligned}
(8.5)\qquad b\,\Delta^{+}(\partial_*) &= (\mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^+_\bullet \mathsf{d}^+_\bullet) - \widehat{\Delta}(\mathsf{d}^+_*))\,\Delta^{+}(\partial_*)\\
&= \Delta^{-}(\mathsf{d}^+_{123}\partial_*) - \widehat{\Delta}(\mathsf{d}^+_*)\,\Delta^{+}(\partial_*)\\
&= -(\widehat{\partial}_1\mathsf{d}^+_1 + \widehat{\partial}_2\mathsf{d}^+_2 + \widehat{\partial}_3\mathsf{d}^+_3)\Delta^{+}(\partial_*) + \Delta^{-}(\mathsf{d}^+_{123}\partial_*)\,,\\
(8.6)\qquad c\,\Delta^{+}(\partial_*) &= (\mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^-_\bullet \mathsf{d}^+_\bullet) - \widehat{\Delta}(\mathsf{d}^-_*))\,\Delta^{+}(\partial_*)\,,\\
(8.7)\qquad \mathsf{d}^-_1\, a\,\Delta^{-}(\partial_*) &= -\mathsf{d}^-_1\, b\,\Delta^{+}(\partial_*)\\
&= \mathsf{d}^-_1\,(\widehat{\Delta}(\mathsf{d}^+_*) - \mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^+_\bullet \mathsf{d}^+_\bullet))\,\Delta^{+}(\partial_*)\,,\\
(8.8)\qquad \widehat{\Delta}(\mathsf{d}^-_*)\,b\,\Delta^{+}(\partial_*) &= \widehat{\Delta}(\mathsf{d}^-_*)\,(\mathsf{C}(\mathsf{d}^-_\bullet \mathsf{d}^+_\bullet \mathsf{d}^+_\bullet) - \widehat{\Delta}(\mathsf{d}^+_*))\,\Delta^{+}(\partial_*)\\
&= -\tfrac{1}{12}\left(D_1^2\,\mathsf{d}^-_1\mathsf{d}^+_1\Delta^{+}(\partial_*) - 6\,D_1\,\mathsf{d}^-_1\Delta^{-}(\mathsf{d}^+_{123}\partial_*)\right)\,.
\end{aligned}
$$

We see that the proposition follows pretty straightforwardly from Theorem 3.20, Corollary 3.21 and Remark 2.7. □

Let us recall that we study the highest singular vectors $\mathbf{v}$ in the generalized Verma modules $M(\mathbf{V})$, $\mathbf{V} = V \otimes T$ where V is an $s\ell(3) \oplus g\ell(1)$-module and T is an $s\ell(2)$-module. Practically it is more convenient to consider both V and T being $\mathfrak{g}_0$-modules with the actions of $s\ell(2)$ on V and $s\ell(3)$ on T being trivial and some action of $g\ell(1)$ on both. The tensor products in (2.4) are then the tensor products of $\mathfrak{g}_0$-modules.

As it was said in Remark 2.3 we suppose that V is an irreducible $\mathfrak{g}_0$-submodule of either $\mathbb{C}[x_1,x_2,x_3]$ or $\mathbb{C}[\,\partial_1,\partial_2,\partial_3]$ with the action of $g\ell(1)$ naturally defined by Remark 2.4. For the module T we take an $(r+1)$-dimensional module with the basic $\{z_+^{r-j} z_-^j[\theta] \mid j = 0,1,\dots,r\}$ and the $\mathfrak{g}_0$-module structure given by the rules

$$
\begin{aligned}
(8.9)\qquad e_3\, z_+^{r-j} z_-^j[\theta] &= j\, z_+^{r-j+1} z_-^{j-1}[\theta]\,,\\
f_3\, z_+^{r-j} z_-^j[\theta] &= (r-j) z_+^{r-j-1} z_-^{j+1}[\theta]\,,\\
(8.10)\qquad Y\, z_+^{r-j} z_-^j[\theta] &= (\theta - r)\, z_+^{r-j} z_-^j[\theta]
\end{aligned}
$$

where $\theta \in \mathbb{C}$ could be arbitrary, and as we said $s\ell(3)$ acts trivially.
To remind the parameters of T *we write* $T = T(r,\theta)$ when it seems needed.

We shall use for $\mathbf{v} \in M(\mathbf{V}) = M(V) \otimes T$ the decomposition

$$
(8.11)\qquad \mathbf{v} = \sum_{j \le j_0} u_j \cdot t_j,
$$

where $u_j \in M(V)$, $t_j = c_j\, z_+^{r-j} z_-^j[\theta] \in T$, and $c_j \in \mathbb{C}$. Let $\mathbf{v}$ be the $\mathfrak{g}_0$-highest weight vector, then it is necessary that $c_0 \neq 0$, and u_0 is the $s\ell(2)$-highest vector of a finite-dimensional $s\ell(2)$-submodule S in $M(V)$.

Suppose $\mathbf{v}$ is also singular. Then from Proposition 2.5 it follows that u_j are the semi-singular weight vectors. Therefore only vectors listed in Propositions 8.1, 8.3 are to be considered for u_j. This is the key idea leading to the classification of singular vectors. The results are summed up as the following main theorem.

Theorem 8.4. *The non-trivial* $\mathfrak{g}_0$*-highest singular vectors* $\mathbf{v}$ *are the following ones (the value of* θ *is written in the square brackets):*

1a. $\mathbf{v} = \mathsf{d}_1^+ x_1^p \cdot z_+^r[0], \quad (\,p, r \geq 0,\ q = 0\,),$

1b. $\mathbf{v} = \mathsf{d}_1^+ x_1^p \cdot z_+^{r-1} z_-[2r+2] - \mathsf{d}_1^- x_1^p \cdot z_+^r[2r+2], \quad (\,p \geq 0,\ r \geq 1,\ q = 0\,),$

2. $\mathbf{v} = \mathsf{d}_1^+ \mathsf{d}_1^- x_1^p \cdot 1[2], \quad (\,p \geq 0,\ q = r = 0\,),$

3. $\mathbf{v} = \mathsf{d}_{123}^+ \cdot z_+^r[-2], \quad (\,p = q = 0,\ r \geq 0),$

4. $\mathbf{v} = a \cdot z_+^{r-3} z_-^3[2r] - b \cdot z_+^{r-2} z_-^2[2r] + c \cdot z_+^{r-1} z_-[2r] - d \cdot z_+^r[2r],$
$(\,p = q = 0,\ r \geq 3\,),$

5. $\mathbf{v} = \mathsf{d}_1^- a \cdot z_-^2[4] - \mathsf{d}_1^- b \cdot z_+ z_-[4] + \mathsf{d}_1^- c \cdot z_+^2[4], \quad (\,p = q = 0,\ r = 2\,),$

6. $\mathbf{v} = \widehat{\Delta}(\mathsf{d}_*^-)\, a \cdot z_-[0] - (d\, a + \widehat{\Delta}(\mathsf{d}_*^-)\, b) \cdot z_+[0], \quad (\,p = q = 0,\ r = 1\,),$

7. $\mathbf{v} = \overset{+}{\Delta}(\partial_*) \partial_3^{q-1} \cdot z_+^r[-2], \quad (\,p = 0,\ q \geq 1,\ r \geq 0\,),$

8. $\mathbf{v} = \overset{+}{\Delta}(\partial_*) \partial_3^{q-1} \cdot z_+^{r-1} z_-[2r] - \overset{-}{\Delta}(\partial_*) \partial_3^{q-1} \cdot z_+^r[2r], \quad (\,p = 0,\ q \geq 1,\ r \geq 1\,),$

9. $\mathbf{v} = \mathsf{d}_1^+ \overset{+}{\Delta}(\partial_*) \cdot z_-[2] - \mathsf{d}_1^+ \overset{-}{\Delta}(\partial_*) \cdot z_+[2], \quad (\,p = 0,\ q = 1,\ r = 1\,),$

10. $\mathbf{v} = \overset{-}{\Delta}(\overset{+}{\Delta}(\partial_*)\partial_*) \partial_3^{q-2} \cdot 1[0], \quad (\,p = 0,\ q \geq 2,\ r = 0\,),$

11. $\mathbf{v} = b \overset{+}{\Delta}(\partial_*) \cdot 1[0], \quad (\,p = 0,\ q = 1,\ r = 0\,).$

Remark 8.5. *Let us compare the list above with the results presented in [4]. In the notations of [4], Theorem 1.2, vectors given in (1a) are of type A, i.e. belong to $M(p, 0; r; y_A)$, where $y_A = \frac{2}{3}p - r$. We only need to notice that according to* (8.10) *and Remark 2.4*

$$Y\, x_1^p \cdot z_+^r[\theta] = \tfrac{2}{3}p - r + \theta.$$

Similarly vectors from (1b) and (2) are of type B, i.e. belong to $M(p, 0; r; y_B)$, where $y_B = \frac{2}{3}p + r + 2$. But from

$$Y\, \partial_3^q \cdot z_+^r[\theta] = -\tfrac{2}{3}q - r + \theta$$

it follows that vectors (7) and (3) are of type C, i.e. are in $M(0, q; r; y_C)$, where $y_C = -\frac{2}{3}q - r - 2$, and the rest, namely (8), (10), (4), (5), (6) and (9) are of type D, i.e. belong to various modules $M(0, q; r; y_D)$, where $y_D = -\frac{2}{3}q + r$.

It is less straightforward to establish the correspondence between explicit expressions for the singular vectors given in [4] and the ones given above. First of all we notice that in [4] we worked with the two versions of a 2-dimensional irreducible representation of $s\ell(2)$, one having the basis $\{z_+, z_-\}$, the other with the basis $\{-\partial_-, \partial_+\}$. Now we need the same spaces, but with the action extended to $s\ell(2) \oplus Y$, therefore the bases are to be written as $\{z_+[\theta], z_+[\theta]\}$, $\{-\partial_-[\theta'], \partial_+[\theta']\}$,

and the representations are isomorphic if and only if $\theta' = \theta + 2$. With this in mind we have no problem indentifying the "general" A, B, C, D cases (1a), (1b), (7), (8) with those of Corollary 2.5 in [4].

Further, our vectors in (2) correspond to those of the case (a) of Corollary 2.8 in [4], and our vectors in (3) correspond to the case (c), in (4) to the case (d) and in (10) to the case (b).

The vector w_1 (given by (2.12) of [4]) is nothing but the vector from (11), up to a sign. The vector w_2, (defined by (2.16) in [4]) is the vector from (5). The vector w_3 (see (2.18) and Lemma 5.26 of [4]) is the vector from (6), and the vector w_4 (given by (2.20) of [4]) is from (9). Thus we have a one-one correspondence between vectors listed in Theorem 8.4 above and in Theorem 2.10 of [4].

Proof. It is clear that Propositions 8.1, 8.3 provide us with few choices for the $s\ell(2)$-module S containing the coefficients u_j of (8.11). Clearly we can suppose that S is irreducible. Because there are only modules S with $\dim S \leq 4$ it is easy to write explicitely possible $s\ell(2)$-highest candidates $\mathbf{v}$ for the singular vectors. It follows from the propositions that the vectors will be semi-singular. Such a vector will be singular if and only if it will satisfy the condition $e_0\,\mathbf{v} = 0$.

We shall study the possible choices and check when the condition is satisfied. Let us remind that

$$\begin{aligned} &[e_0,\, \mathsf{d}_1^-] = -2f_3, \qquad &&[e_0,\, \mathsf{d}_1^+] = h_0,\\ &[e_0,\, \mathsf{d}_2^-] = 0, &&[e_0,\, \mathsf{d}_2^+] = f_1,\\ &[e_0,\, \mathsf{d}_3^-] = 0, &&[e_0,\, \mathsf{d}_3^+] = -f_{12}, \end{aligned} \tag{8.12}$$

and $[e_0,\, \widehat{\partial}_1] = 0$, $[e_0,\, \widehat{\partial}_2] = \mathsf{d}_3^-$, $[e_0,\, \widehat{\partial}_3] = -\mathsf{d}_2^-$.

1) $S = \langle\, \mathsf{d}_1^+ x_1^p, \mathsf{d}_1^- x_1^p \,\rangle$.
Here either $\mathbf{v} = \mathsf{d}_1^+ x_1^p \cdot z_+^r[\theta]$, or $\mathbf{v} = \mathsf{d}_1^+ x_1^p \cdot z_+^{r-1} z_-[\theta] - \mathsf{d}_1^- x_1^p \cdot z_+^r[\theta]$. From the condition $e_0\,\mathbf{v} = 0$ it follows $\theta = 0$ for the former which gives us vectors from (*1a*) . For the latter, we compute

$$\begin{aligned} e_0\mathbf{v} &= e_0(\mathsf{d}_1^+ x_1^p \cdot z_+^{r-1} z_-[\theta] - \mathsf{d}_1^- x_1^p \cdot z_+^r[\theta]\,)\\ &= h_0\, x_1^p \cdot z_+^{r-1} z_-[\theta] + 2f_3\, x_1^p \cdot z_+^r[\theta]\\ &= (2-\theta) x_1^p \cdot z_+^{r-1} z_-[\theta] + 2r\, x_1^p \cdot z_+^{r-1} z_-[\theta]\\ &= (2-\theta+2r) x_1^p \cdot z_+^{r-1} z_-[\theta], \end{aligned}$$

hence $\theta = 2 + 2r$ which gives us vectors from (*1b*) of the theorem.

2) $S = \langle\, \mathsf{d}_1^- \mathsf{d}_1^+ x_1^p \,\rangle$.
We come to $\mathbf{v} = \mathsf{d}_1^- \mathsf{d}_1^+ x_1^p \cdot z_+^r[\theta]$. Now

$$\begin{aligned} e_0\, \mathsf{d}_1^- \mathsf{d}_1^+ x_1^p \cdot z_+^r[\theta] &= (-2f_3)\mathsf{d}_1^+ x_1^p \cdot z_+^r[\theta] - \mathsf{d}_1^-(h_0\, x_1^p \cdot z_+^r[\theta])\\ &= -2\mathsf{d}_1^- x_1^p \cdot z_+^r[\theta] - 2r\mathsf{d}_1^+ x_1^p \cdot z_+^{r-1} z_-[\theta] + \theta\, \mathsf{d}_1^- x_1^p \cdot z_+^r[\theta], \end{aligned}$$

hence $\theta = 0$, $r = 0$. We come to the vector from (*2*).

3) $S = \langle\, a,\, b,\, c,\, d\,\rangle$ (in the notations of Proposition 8.1).
We have four choices for $\mathbf{v}$.

$$\begin{aligned}
\mathbf{v}_1 &= a\cdot z_+^r[\theta],\\
\mathbf{v}_2 &= b\cdot z_+^r[\theta] - 3a\cdot z_+^{r-1}z_-[\theta],\\
\mathbf{v}_3 &= c\cdot z_+^r[\theta] - 2b\cdot z_+^{r-1}z_-[\theta] + 3a\cdot z_+^{r-2}z_-^2[\theta],\\
\mathbf{v}_4 &= d\cdot z_+^r[\theta] - c\cdot z_+^{r-1}z_-[\theta] + b\cdot z_+^{r-2}z_-^2[\theta] - a\cdot z_+^{r-3}z_-^3[\theta].
\end{aligned}$$

From (8.12) it follows

$$\begin{aligned}
[e_0,\, a] \;=\; e_0\,\mathsf{d}_1^+\mathsf{d}_2^+\mathsf{d}_3^+ &= h_0\mathsf{d}_2^+\mathsf{d}_3^+ - \mathsf{d}_1^+f_1\mathsf{d}_3^+ + \mathsf{d}_1^+\mathsf{d}_2^+f_{12}\\
&= \mathsf{d}_2^+\mathsf{d}_3^+(h_0-2) - \mathsf{d}_1^+\mathsf{d}_3^+f_1 + \mathsf{d}_1^+\mathsf{d}_2^+f_{12}.
\end{aligned} \tag{8.13}$$

We get immediately that $e_0\,\mathbf{v}_1 = (-\theta-2)\mathsf{d}_2^+\mathsf{d}_3^+\cdot z_+^r[\theta]$, hence $\theta=-2$ and we come to the vector from (*3*).

Notice that $[e_0,\, f_3]=0$. This makes it easy to calculate from (8.13) that

$$\begin{aligned}
[e_0,\, b] = (\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)(h_0-2) + \mathsf{d}_2^+\mathsf{d}_3^+(-2f_3) - (\mathsf{d}_1^-\mathsf{d}_3^+ + \mathsf{d}_1^+\mathsf{d}_3^+)f_1+\\
+ (\mathsf{d}_1^-\mathsf{d}_2^+ + \mathsf{d}_1^+\mathsf{d}_2^-)f_{12}.
\end{aligned} \tag{8.14}$$

Therefore

$$e_0\,\mathbf{v}_2 = (-\theta-2)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)\cdot z_+^r[\theta] + (-2r+3\theta)\mathsf{d}_2^+\mathsf{d}_3^+\cdot z_+^{r-1}z_-[\theta].$$

We come to equations $\theta=-2$ and $2r=3\theta$ that gives $r=-3$ which is impossible.

Similarly

$$[e_0,\, c] = \mathsf{d}_2^-\mathsf{d}_3^-(h_0-2) + (\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)(-2f_3) - \mathsf{d}_1^-\mathsf{d}_3^-f_1 + \mathsf{d}_1^-\mathsf{d}_2^-f_{12}. \tag{8.15}$$

Formulae (8.13), (8.14) and (8.15) imply that

$$\begin{aligned}
e_0\,\mathbf{v}_3 \;=\; &(-\theta-2)\mathsf{d}_2^-\mathsf{d}_3^-\cdot z_+^r[\theta]\\
&+ (-2r+3\theta)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)\cdot z_+^{r-1}z_-[\theta]\\
&+ (4(r-1)r + 3(2-\theta))\mathsf{d}_2^+\mathsf{d}_3^+\cdot z_+^{r-2}z_-^2[\theta].
\end{aligned}$$

We see that it should be $\theta=2$ and $r=\theta$ which give the impossible value $r=-2$.

For d we immediately get from (8.12)

$$[e_0,\, d] = \mathsf{d}_2^-\mathsf{d}_3^-(-2f_3). \tag{8.16}$$

Therefore

$$\begin{aligned}
e_0\,\mathbf{v}_4 \;=\; &(-2r+\theta)\mathsf{d}_2^-\mathsf{d}_3^-\cdot z_+^{r-1}z_-[\theta] + (2(r-1)+(2-\theta))(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)\cdot z_+^{r-2}z_-^2[\theta]\\
&+ (-2(r-2)-(4-\theta))\mathsf{d}_2^+\mathsf{d}_3^+\cdot z_+^{r-3}z_-^3[\theta].
\end{aligned}$$

We conclude that $\theta=2r$, $r\geq 3$ and this is what stated in (*4*).

4) $S = \langle \mathsf{d}_1^- a,\, \mathsf{d}_1^- b,\, \mathsf{d}_1^- c \rangle$.
From (8.1), (8.2) we conclude that there are three choices for $\mathbf{v}$.

$$\begin{aligned}
\mathbf{v}_1 &= \mathsf{d}_1^- a \cdot z_+^r[\theta], \\
\mathbf{v}_2 &= \mathsf{d}_1^- b \cdot z_+^r[\theta] - 2\mathsf{d}_1^- a \cdot z_+^{r-1} z_-[\theta], \\
\mathbf{v}_3 &= \mathsf{d}_1^- c \cdot z_+^r[\theta] - \mathsf{d}_1^- b \cdot z_+^{r-1} z_-[\theta] + \mathsf{d}_1^- a \cdot z_+^{r-2} z_-^2[\theta].
\end{aligned}$$

On the other hand $[e_0,\, \mathsf{d}_1^- x] = (-2f_3)x - \mathsf{d}_1^-[e_0,\, x]$. Then using (8.13) we easily calculate that

$$\begin{aligned}
e_0\, \mathbf{v}_1 =& \ (-2f_3)a \cdot z_+^r[\theta] - \mathsf{d}_1^-(e_0\, a \cdot z_+^r[\theta]) \\
=& -2b \cdot z_+^r[\theta] - 2r\, a \cdot z_+^{r-1} z_-[\theta] - (-\theta - 2)\mathsf{d}_1^- \mathsf{d}_2^+ \mathsf{d}_3^+ \cdot z_+^r[\theta] \neq 0.
\end{aligned}$$

Thus $\mathbf{v}_1$ is never singular.
Similarly for $\mathbf{v}_2$ we get

$$\begin{aligned}
e_0\, \mathbf{v}_2 =& (-4)c \cdot z_+^r[\theta] + (-2r+4)b \cdot z_+^{r-1} z_-[\theta] + 4(r-1)a \cdot z_+^{r-2} z_-^2[\theta] \\
&+ (\theta+2)(\mathsf{d}_1^- \mathsf{d}_2^- \mathsf{d}_3^+ + \mathsf{d}_1^- \mathsf{d}_2^+ \mathsf{d}_3^-) \cdot z_+^r[\theta] + (2r - 2\theta)\mathsf{d}_1^- \mathsf{d}_2^+ \mathsf{d}_3^+ \cdot z_+^{r-1} z_-[\theta].
\end{aligned}$$

Also it is never zero, we get no singular vector.
For the last one

$$\begin{aligned}
e_0\, \mathbf{v}_3 =& \ (\theta-4)d \cdot z_+^r[\theta] - 2(r-2)(c \cdot z_+^{r-1} z_-[\theta] - b \cdot z_+^{r-2} z_-^2[\theta] + a \cdot z_+^{r-3} z_-^3[\theta]) \\
&+ (2r-\theta)(\mathsf{d}_1^- \mathsf{d}_2^- \mathsf{d}_3^+ + \mathsf{d}_1^- \mathsf{d}_2^+ \mathsf{d}_3^-) \cdot z_+^{r-1} z_-[\theta] + (-2r+\theta)\mathsf{d}_1^- \mathsf{d}_2^+ \mathsf{d}_3^+ \cdot z_+^{r-2} z_-^2[\theta].
\end{aligned}$$

We get zero only when $r = 2$, $\theta = 4$ which gives us the singular vector from (*5*).

5) $S = \langle \widehat{\Delta}(\mathsf{d}_*^-)a,\, \widehat{\Delta}(\mathsf{d}_*^-)b,\, \widehat{\Delta}(\mathsf{d}_*^-)c \rangle \oplus \langle d\, a - a\, d \rangle$.
We have here the following choices for $\mathbf{v}$.

$$\begin{aligned}
\mathbf{v}_1 &= \widehat{\Delta}(\mathsf{d}_*^-)a \cdot z_+^r[\theta], \\
\mathbf{v}_2 &= \widehat{\Delta}(\mathsf{d}_*^-)(b \cdot z_+^r[\theta] - 2a \cdot z_+^{r-1} z_-[\theta]), \\
\mathbf{v}_3 &= (d\, a - a\, d) \cdot z_+^r[\theta] = (2\, d\, a + \widehat{\Delta}(\mathsf{d}_*^-)\, b) \cdot z_+^r[\theta], \\
\mathbf{v}_4 &= \widehat{\Delta}(\mathsf{d}_*^-)(c \cdot z_+^r[\theta] - b \cdot z_+^{r-1} z_-[\theta] + a \cdot z_+^{r-2} z_-^2[\theta]),
\end{aligned}$$

where $\mathbf{v}_2$ and $\mathbf{v}_3$ have the same weight, so we have check their linear combinations as well.
First of all

$$e_0\, \widehat{\Delta}(\mathsf{d}_*^-) = -2\, \mathsf{d}_2^- \mathsf{d}_3^- - 2\, \widehat{\partial}_1\, f_3 - \widehat{\Delta}(\mathsf{d}_*^-)\, e_0\,. \tag{8.17}$$

Now from (8.13) we get

$$\begin{aligned}
e_0\, \mathbf{v}_1 =& -2\, \mathsf{d}_2^- \mathsf{d}_3^-\, a \cdot z_+^r[\theta] - 2\widehat{\partial}_1(b \cdot z_+^r[\theta] - r\, a \cdot z_+^{r-1} z_-[\theta]) \\
&+ (\theta+2)\widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_2^+ \mathsf{d}_3^+ \cdot z_+^r[\theta],
\end{aligned}$$

that is always non-zero.

Now we shall calculate the left hand side of $e_0\,\mathbf{v}_2 = 0$. Similarly from (8.17) we conclude that the only term containing $z_+^{r-2}z_-^2[\theta]$ is $4(r-1)\widehat{\partial}_1 a\cdot z_+^{r-2}z_-^2[\theta]$. At the same time

$$\begin{aligned} e_0\,da\cdot z_+^r[\theta] &= \mathsf{d}_2^-\mathsf{d}_3^-\,(-2f_3)a\cdot z_+^r[\theta] - d\,e_0 a\cdot z_+^r[\theta]\\ &= -2\mathsf{d}_2^-\mathsf{d}_3^- b\cdot z_+^r[\theta] - 2r\mathsf{d}_2^-\mathsf{d}_3^- a\cdot z_+^{r-1}z_-[\theta] + (\theta+2)\,d\,\mathsf{d}_2^+\mathsf{d}_3^+\cdot z_+^r[\theta]\,. \end{aligned}$$

This implies that no term in $e_0\,\mathbf{v}_3$ contains $z_+^{r-2}z_-^2[\theta]$. Therefore unless $r=1$ we have got $e_0(\alpha\mathbf{v}_2+\beta\mathbf{v}_3)\neq 0$.

Now let $r=1$, then

$$\begin{aligned} e_0\,\widehat{\Delta}(\mathsf{d}_*^-)a\cdot z_-[\theta] &= -\,2\mathsf{d}_2^-\mathsf{d}_3^-\,a\cdot z_-[\theta] - 2\widehat{\partial}_1 b\cdot z_-[\theta]\\ &\qquad + \theta\widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_2^+\mathsf{d}_3^+\cdot z_-[\theta]\,,\\ e_0\,\widehat{\Delta}(\mathsf{d}_*^-)b\cdot z_+[\theta] &= -\,2\mathsf{d}_2^-\mathsf{d}_3^-\,b\cdot z_+[\theta] - 2\widehat{\partial}_1 b\cdot z_-[\theta] - 4\widehat{\partial}_1 c\cdot z_+[\theta]\\ &\qquad + \theta\widehat{\Delta}(\mathsf{d}_*^-)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)\cdot z_+[\theta] + 2\widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_2^+\mathsf{d}_3^+\cdot z_-[\theta]\,. \end{aligned}$$

It is not difficult to conclude that only when $\theta=2$ the vector $\mathbf{v}=\mathbf{v}_2+\mathbf{v}_3$ could be singular. This vector is indeed singular (equal to w_3 in the notations of [4]), and this leads us to the case (*6*) of the theorem.

6) $S=\langle \mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^-)a,\ \mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^-)b\rangle$.
We have to work with vectors

$$\begin{aligned} \mathbf{v}_1 &= \mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^-)a\cdot z_+^r[\theta],\\ \mathbf{v}_2 &= \mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^-)(b\cdot z_+^r[\theta] - a\cdot z_+^{r-1}z_-[\theta])\,. \end{aligned}$$

For $\mathbf{v}_1$ it is clear that

$$e_0\mathbf{v}_1 = (\dots)\cdot z_+^r[\theta] - 2r(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)a\cdot z_+^{r-1}z_-[\theta]\,,$$

hence we are left with $r=0$. Then

$$e_0\mathbf{v}_1 = (2da - 2\widehat{\Delta}(\mathsf{d}_*^-)b + 2\mathsf{d}_1^-\widehat{\partial}_1 b - (\theta+2)\mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^-)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-))\cdot z_+[\theta]\,,$$

that never gives zero (da could not cancel with other terms).

For $\mathbf{v}_2$ again we can first compute the coefficient by $z_+^{r-2}z_-^2[\theta]$. The computation gives $-2(r-1)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)a$ so $r=1$. Then we can calculate the the coefficient by $z_+^{r-1}z_-[\theta]$. Here we get

$$-da + (2r+2-\theta)\mathsf{d}_1^-\widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_2^+\mathsf{d}_3^+ \neq 0\,,$$

hence $\mathbf{v}_2$ is not singular as well.

From now on we consider vectors of Proposition 8.3.

7) $S=\langle \Delta^+(\partial_*)\partial_3^{q-1}\,, \Delta^-(\partial_*)\partial_3^{q-1}\rangle$.
We are to look at

$$\begin{aligned} \mathbf{v}_1 &= \Delta^+(\partial_*)\partial_3^{q-1}\cdot z_+^r[\theta]\,,\\ \mathbf{v}_2 &= \Delta^-(\partial_*)\partial_3^{q-1}\cdot z_+^r[\theta] - \Delta^+(\partial_*)\partial_3^{q-1}\cdot z_+^{r-1}z_-[\theta]\,. \end{aligned}$$

Let us notice that

$$[e_0,\, \Delta^{+}(\partial_*)] = \partial_1(h_0-2)+\partial_2 f_1+\partial_3 f_{12}\,, \tag{8.18}$$

$$[e_0,\, \Delta^{-}(\partial_*)] = -\,2\partial_1 f_3\,. \tag{8.19}$$

Now it is easy to calculate that

$$e_0\mathbf{v}_1 = ((q-1)-\theta-2-(q-1))\partial_1\partial_3^{q-1}\cdot z_+^r[\theta]\,,$$

thus $\theta=-2$ for any $q\geq 1$, and we come to sigular vectors from (*7*).

Similarly

$$e_0\mathbf{v}_2 = (-2r+\theta)\partial_1\partial_3^{q-2}\cdot z_+^{r-1}z_-[\theta]\,,$$

hence $\theta=2r$, and we come to sigular vectors from (*8*).

8) $S=\langle \mathsf{d}_1^+\Delta^{+}(\partial_*)\,,\mathsf{d}_1^+\Delta^{-}(\partial_*),\, \mathsf{d}_1^-\Delta^{+}(\partial_*)\,,\mathsf{d}_1^-\Delta^{-}(\partial_*)\rangle$.
To simplify formulae we shall write just Δ^{+}, Δ^{-} instead of $\Delta^{+}(\partial_*)$, $\Delta^{-}(\partial_*)$ whenever it does not lead to misunderstanding.

For any $t\in T$ we have

$$\begin{aligned}
e_0\,\mathsf{d}_1^+\Delta^{+}\cdot t &= \quad\;\; \Delta^{+}\cdot h_0 t-\mathsf{d}_1^+\partial_1\cdot(h_0-2)t\,,\\
e_0\,\mathsf{d}_1^+\Delta^{-}\cdot t &= \quad\;\; \Delta^{-}\cdot(h_0+2)t+2\mathsf{d}_1^+\partial_1\cdot f_3 t\,,\\
e_0\,\mathsf{d}_1^-\Delta^{+}\cdot t &= -\,2\,\Delta^{-}\cdot t-\mathsf{d}_1^-\partial_1\cdot(h_0-2)t-2\Delta^{+}\cdot f_3 t\,,\\
e_0\,\mathsf{d}_1^-\Delta^{-}\cdot t &= -\,2\,(\Delta^{-}-\mathsf{d}_1^-\partial_1)\cdot f_3 t\,,
\end{aligned}$$

Let us notice that the $s\ell(2)$-module S is a direct sum of a 3-dimensional irreducible one and 1-dimensional one. The latter is generated by $\mathsf{d}_1^-\Delta^{+}(\partial_*)-\mathsf{d}_1^+\Delta^{-}(\partial_*)$. Therefore we have to consider the following vectors

$$\begin{aligned}
\mathbf{v}_1 &= \mathsf{d}_1^+\Delta^{+}\cdot z_+^r[\theta]\,,\\
\mathbf{v}_2 &= (\mathsf{d}_1^+\Delta^{-}+\mathsf{d}_1^-\Delta^{+})\cdot z_+^r[\theta]-2\mathsf{d}_1^+\Delta^{+}\cdot z_+^{r-1}z_-[\theta]\,,\\
\mathbf{v}_3 &= (\mathsf{d}_1^+\Delta^{-}-\mathsf{d}_1^-\Delta^{+})\cdot z_+^r[\theta]\,,\\
\mathbf{v}_4 &= \mathsf{d}_1^-\Delta^{-}\cdot z_+^r[\theta]-(\mathsf{d}_1^+\Delta^{-}+\mathsf{d}_1^-\Delta^{+})\cdot z_+^{r-1}z_-[\theta]+\mathsf{d}_1^+\Delta^{+}\cdot z_+^{r-2}z_-^2[\theta]\,,
\end{aligned}$$

and the linear combinations of $\mathbf{v}_2$, $\mathbf{v}_3$ as well.

Starting with $\mathbf{v}_1$ we get

$$e_0\mathbf{v}_1 = -\,\theta\Delta^{+}\cdot z_+^r[\theta]+(\theta+2)\mathsf{d}_1^+\partial_1\cdot z_+^r[\theta]\,.$$

For the vector to be singular it should be $\theta=\theta+2=0$, which is impossible.

Now from (8.18), (8.19) we see that

$$\begin{aligned}
e_0(\mathsf{d}_1^+\Delta^{-}\cdot z_+^r[\theta]) &= (-\theta+2)\Delta^{-}\cdot z_+^r[\theta]+2r\,\mathsf{d}_1^+\partial_1\cdot z_+^{r-1}z_-[\theta]\,,\\
e_0(\mathsf{d}_1^-\Delta^{+}\cdot z_+^r[\theta]) &= -\,2\,\Delta^{-}\cdot z_+^r[\theta]+(\theta+2)\mathsf{d}_1^-\partial_1\cdot z_+^r[\theta]-2r\Delta^{+}\cdot z_+^{r-1}z_-[\theta]\,,\\
e_0(-2\mathsf{d}_1^+\Delta^{+}\cdot z_+^{r-1}z_-[\theta]) &= (2\theta-4)\Delta^{+}\cdot z_+^{r-1}z_-[\theta]-2\theta\,\mathsf{d}_1^+\partial_1\cdot z_+^{r-1}z_-[\theta]\,.
\end{aligned}$$

It follows that $e_0\,(\alpha\mathbf{v}_2+\beta\mathbf{v}_3)=0$ if and only if

$$\begin{aligned}\alpha(-\theta)+\beta(2-\theta+2) &= 0\,,\\ (\alpha-\beta)(\theta+2) &= 0\,,\\ \alpha(2\theta-4-2r)+\beta\,2r &= 0\,,\\ \alpha(-2\theta+2r)+\beta\,2r &= 0\,.\end{aligned}$$

From the second equation we see that either $\alpha-\beta=0$ or $\theta=-2$. It is easy to solve the system in each case. We get only one solution with $r=1$, $\theta=2$, $\alpha=\beta$ and that gives us the vector from (*9*).

For the last vector $\mathbf{v}_4$ we calculate

$$\begin{aligned}
e_0\,\mathsf{d}_1^-\Delta^-\cdot z_+^r[\theta] &= -\,2r\,\Delta^-\cdot z_+^{r-1}z_-[\theta]+2r\,\mathsf{d}_1^-\partial_1\cdot z_+^{r-1}z_-[\theta]\,,\\
e_0\,\mathsf{d}_1^-\Delta^+\cdot z_+^{r-1}z_-[\theta] &= -\,2\,\Delta^-\cdot z_+^{r-1}z_-[\theta]+\theta\,\mathsf{d}_1^-\partial_1\cdot z_+^{r-1}z_-[\theta]-2(r-1)\,\Delta^+\cdot z_+^{r-2}z_-^2[\theta]\,,\\
e_0\,\mathsf{d}_1^+\Delta^-\cdot z_+^{r-1}z_-[\theta] &= -\,\theta\,\Delta^-\cdot z_+^{r-1}z_-[\theta]+2(r-1)\,\mathsf{d}_1^+\partial_1\cdot z_+^{r-2}z_-^2[\theta]\,,\\
e_0\,\mathsf{d}_1^+\Delta^+\cdot z_+^{r-2}z_-^2[\theta] &= (4-\theta)\Delta^+\cdot z_+^{r-2}z_-^2[\theta]-(2-\theta)\,\mathsf{d}_1^+\partial_1\cdot z_+^{r-2}z_-^2[\theta]\,.
\end{aligned}$$

and combine them to get an explicite expression for $e_0\,\mathbf{v}_4=0$. We come to four linear equations on r, θ, that are inconsistent. Thus no singular vector of this form exists.

9) $S=\langle\Delta^-(\Delta^+(\partial_*)\partial_*)\rangle$.
Here we are to compute

$$\begin{aligned}
e_0\,\Delta^-(\Delta^+(\partial_*)\partial_*)\cdot z_+^r[\theta] &= -\,2f_3\,\Delta^+\partial_1\cdot z_+^r[\theta]-\sum\mathsf{d}_i^-\,e_0\Delta^+\partial_i\cdot z_+^r[\theta]\\
&= -\,2\,\Delta^-\partial_1\cdot z_+^r[\theta]-2r\,\Delta^+\partial_1\cdot z_+^{r-1}z_-[\theta]\\
&\quad+\big((\theta+2)\mathsf{d}_1^-\partial_1^2+\mathsf{d}_2^-((3+\theta)\partial_1\partial_2-\partial_2\partial_1)+\mathsf{d}_3^-((3+\theta)\partial_1\partial_3-\partial_3\partial_1)\big)\cdot z_+^r[\theta]\\
&= \theta\,\Delta^-\partial_1\cdot z_+^r[\theta]-2r\,\Delta^+\partial_1\cdot z_+^{r-1}z_-[\theta]\,.
\end{aligned}$$

We come to conditions $r=0$, $\theta=0$ and then to the singular vector listed in (*10*).

10) $S=\langle\mathsf{d}_1^-\mathsf{d}_1^+\Delta^+(\partial_*)\rangle$.
Similarly we are to calculate

$$\begin{aligned}
e_0\,\mathsf{d}_1^-\mathsf{d}_1^+\Delta^+(\partial_*)\cdot z_+^r[\theta] &= \Big(-2f_3\,\mathsf{d}_1^+\Delta^+(\partial_*)-\mathsf{d}_1^-\,h_0\Delta^+(\partial_*)+\mathsf{d}_1^-\mathsf{d}_1^+\,e_0\Delta^+(\partial_*)\Big)\cdot z_+^r[\theta]\\
&= \Big((\theta-2)\mathsf{d}_1^-\Delta^+(\partial_*)-2\mathsf{d}_1^+\Delta^-(\partial_*)-(\theta+2)\mathsf{d}_1^-\mathsf{d}_1^+\Big)\cdot z_+^r[\theta]\\
&\quad-2r\,\mathsf{d}_1^+\Delta^+(\partial_*)\cdot z_+^{r-1}z_-[\theta]\,\neq 0\,.
\end{aligned}$$

No singular vector exists.

11) $S=\langle b\Delta^+,\,c\Delta^+,\,d\Delta^+\rangle$.
Let us use the fact that $a\Delta^+=0$, thus $b\Delta^++a\Delta^-=0$, $c\Delta^++b\Delta^-=0$, and $d\Delta^++c\Delta^-=0$.

Therefore can consider vectors in the form

$$\begin{aligned}
\mathbf{v}_1 &= a\Delta^{-} \cdot z_{+}^{r}[\theta]\,,\\
\mathbf{v}_2 &= b\Delta^{-} \cdot z_{+}^{r}[\theta] - 2a\Delta^{-} \cdot z_{+}^{r-1} z_{-}[\theta]\,,\\
\mathbf{v}_3 &= c\Delta^{-} \cdot z_{+}^{r}[\theta] - b\Delta^{-} \cdot z_{+}^{r-1} z_{-}[\theta] + a\Delta^{-} \cdot z_{+}^{r-2} z_{-}^{2}[\theta]\,.
\end{aligned}$$

Now for $\mathbf{v}_1$, we use (8.13), (8.19) and get

$$\begin{aligned}
e_0\,\mathbf{v}_1 &= \Big([e_0,\,a]\Delta^{+} - a\,e_0\Delta^{-}\Big) \cdot z_{+}^{r}[\theta]\\
&= \Big(\mathsf{d}_2^{+}\mathsf{d}_3^{+}(h_0-2)\Delta^{-} - a(-2f_3)\partial_1\Big) \cdot z_{+}^{r}[\theta]\\
&= -\,\theta\mathsf{d}_2^{+}\mathsf{d}_3^{+}\Delta^{-} \cdot z_{+}^{r}[\theta] + 2r\,a\partial_1 \cdot z_{+}^{r-1} z_{-}[\theta]\,.
\end{aligned}$$

We conclude that $r=0$, $\theta=0$ and this gives us the vector from (*11*).

Similarly

$$\begin{aligned}
e_0\,\mathbf{v}_2 = &\Big([e_0,\,b]\Delta^{+} - b\,e_0\Delta^{-}\Big) \cdot z_{+}^{r}[\theta] - 2\Big([e_0,\,a]\Delta^{+} - a\,e_0\Delta^{-}\Big) \cdot z_{+}^{r-1} z_{-}[\theta]\\
= &\Big(-\theta(\mathsf{d}_2^{-}\mathsf{d}_3^{+} + \mathsf{d}_2^{+}\mathsf{d}_3^{-})\Delta^{-}\Big) \cdot z_{+}^{r}[\theta]\\
&+ \Big((-2r+2\theta-4)\mathsf{d}_2^{+}\mathsf{d}_3^{+}\Delta^{-} + 2r\,b\partial_1\Big) \cdot z_{+}^{r-1} z_{-}[\theta]\\
&- 4(r-1)a\partial_1 \cdot z_{+}^{r-2} z_{-}^{2}[\theta]\,.
\end{aligned}$$

We see that $e_0\,\mathbf{v}_2 \neq 0$ whatever r, θ.

Again by the same argument we have

$$\begin{aligned}
e_0\,\mathbf{v}_3 = &\Big([e_0,\,c]\Delta^{+} - c\,e_0\Delta^{-}\Big) \cdot z_{+}^{r}[\theta] - \Big([e_0,\,b]\Delta^{+} - b\,e_0\Delta^{-}\Big) \cdot z_{+}^{r-1} z_{-}[\theta]\\
&+ \Big([e_0,\,a]\Delta^{+} - a\,e_0\Delta^{-}\Big) \cdot z_{+}^{r-2} z_{-}^{2}[\theta]\,.
\end{aligned}$$

By $z_{+}^{r}[\theta]$ we get the coefficient $\theta\,\mathsf{d}_2^{-}\mathsf{d}_3^{-}\Delta^{-}$, thus $\theta=0$. The coefficient by $z_{+}^{r-1}z_{-}[\theta]$ gives

$$(-2r-(4-\theta-2))(\mathsf{d}_2^{-}\mathsf{d}_3^{+} + \mathsf{d}_2^{+}\mathsf{d}_3^{-})\Delta^{-} + 2r\,c\partial_1 \;=\; 0\,,$$

and from the coefficient by $z_{+}^{r-2}z_{-}^{2}[\theta]$ it comes

$$(2(r-1)+(6-\theta-2))\mathsf{d}_2^{+}\mathsf{d}_3^{+}\Delta^{-} - 2(r-1)\,b\partial_1 \;=\; 0\,.$$

Clearly no singular vectors appear.

12) $S = \langle \mathsf{d}_1^{-}\mathsf{d}_1^{+}\Delta^{-}(\Delta^{+}(\partial_*)\partial_*)\rangle$.
Here

$$\mathbf{v} \;=\; \mathsf{d}_1^{-}\mathsf{d}_1^{+}\Delta^{-}(\Delta^{+}(\partial_*)\partial_*) \cdot z_{+}^{r}[\theta]\,,$$

It is enough to compute the coefficient by $z_+^r[\theta]$ in $e_0\mathbf{v}$ in order to realize that the result is non-zero. Namely

$$
\begin{aligned}
e_0\,\mathbf{v} &= \Big(-2f_3\mathsf{d}_1^+\Delta^-(\Delta^+(\partial_*)\partial_*) - \mathsf{d}_1^-\,h_0\Delta^-(\Delta^+(\partial_*)\partial_*) + \mathsf{d}_1^-\mathsf{d}_1^+\,e_0\Delta^-(\Delta^+(\partial_*)\partial_*)\Big)\cdot z_+^r[\theta] \\
&= \Big((\theta-4)\mathsf{d}_1^-\Delta^-(\Delta^+(\partial_*)\partial_*) + \theta\mathsf{d}_1^-\mathsf{d}_1^+\Delta^-\partial_1\Big)\cdot z_+^r[\theta] + \ldots \neq 0\,.
\end{aligned}
$$

The vector is never singular.

13) $S = \langle \mathsf{d}_1^- a\Delta^-, \mathsf{d}_1^- b\Delta^-\rangle$.
Now

$$
\begin{aligned}
\mathbf{v}_1 &= \mathsf{d}_1^- a\Delta^-\cdot z_+^r[\theta]\,, \\
\mathbf{v}_2 &= \mathsf{d}_1^- b\Delta^-\cdot z_+^r[\theta] - \mathsf{d}_1^- a\Delta^-\cdot z_+^{r-1}z_-[\theta]\,.
\end{aligned}
$$

Immediately we get

$$
\begin{aligned}
e_0\,\mathbf{v}_1 &= \Big((-2f_3)a\Delta^- - \mathsf{d}_1^-[e_0,\,a]\Delta^- + \mathsf{d}_1^- a\partial_1(-2f_3)\Big)\cdot z_+^r[\theta]\,, \\
&= (\theta\mathsf{d}_1^-\mathsf{d}_2^+\mathsf{d}_3^+ - 2\,b)\Delta^-\cdot z_+^r[\theta] + \ldots \neq 0\,.
\end{aligned}
$$

Also

$$
\begin{aligned}
e_0\,\mathbf{v}_2 &= \Big((-2f_3)b\Delta^- - \mathsf{d}_1^-[e_0,\,b]\Delta^- + \mathsf{d}_1^- b\partial_1(-2f_3)\Big)\cdot z_+^r[\theta] + \ldots \\
&= (\theta\mathsf{d}_1^-(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-) - 4c)\Delta^-\cdot z_+^r[\theta] + \ldots \neq 0\,.
\end{aligned}
$$

There are no singular vectors here.

14) $S = \langle \widehat{\Delta}(\mathsf{d}_*^-)b\Delta^+,\, \widehat{\Delta}(\mathsf{d}_*^-)c\Delta^+\rangle$.
Again we can rewrite the basis using the relations $b\Delta^+ + a\Delta^- = 0$, $c\Delta^+ + b\Delta^- = 0$. Thus we are to consider vectors

$$
\begin{aligned}
\mathbf{v}_1 &= \widehat{\Delta}(\mathsf{d}_*^-)a\Delta^-\cdot z_+^r[\theta]\,, \\
\mathbf{v}_2 &= \widehat{\Delta}(\mathsf{d}_*^-)b\Delta^-\cdot z_+^r[\theta] - \widehat{\Delta}(\mathsf{d}_*^-)a\Delta^-\cdot z_+^{r-1}z_-[\theta]\,.
\end{aligned}
$$

Here (8.17), (8.13) and (8.19) show that

$$
\begin{aligned}
e_0\,\mathbf{v}_1 &= \Big((-2\mathsf{d}_2^-\mathsf{d}_3^- - 2\widehat{\partial}_1 f_3)a\Delta^- - \widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_2^+\mathsf{d}_3^+(h_0-2)\Delta^- - 2\widehat{\Delta}(\mathsf{d}_*^-)a\partial_1 f_3\Big)\cdot z_+^r[\theta] \\
&= \Big(-2\mathsf{d}_2^-\mathsf{d}_3^-\,a\Delta^- - 2\widehat{\partial}_1 b\Delta^- + \theta\widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_2^+\mathsf{d}_3^+\Delta^-\Big)\cdot z_+^r[\theta] + \ldots \\
&= \Big(2\mathsf{d}_2^-\mathsf{d}_3^-\,b\Delta^+ + 2\widehat{\partial}_1 c\Delta^+ + \theta\widehat{\Delta}(\mathsf{d}_*^-)\mathsf{d}_2^+\mathsf{d}_3^+\Delta^-\Big)\cdot z_+^r[\theta] + \ldots\,.
\end{aligned}
$$

It is clear that when we put the terms on the right hand side in order, we get only one term not containing $\widehat{\partial}_1, \widehat{\partial}_2, \widehat{\partial}_3$, namely $2\mathsf{d}_2^-\mathsf{d}_3^-\mathsf{d}_1^-\mathsf{d}_2^+\mathsf{d}_3^+\mathsf{d}_1^+\partial_1\cdot z_+^r[\theta]$, hence $e_0\,\mathbf{v}_1 \neq 0$.

Calculating with $e_0\,\mathbf{v}_2$ we shall check only terms with $z_+^r[\theta]$. Then

$$
\begin{aligned}
e_0\,\mathbf{v}_2 &= \Big((-2\mathsf{d}_2^-\mathsf{d}_3^- - 2\widehat{\partial}_1 f_3)b\Delta^- - \widehat{\Delta}(\mathsf{d}_*^-)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)(h_0-2)\Delta^-\Big)\cdot z_+^r[\theta] + \ldots \\
&= \Big(2\mathsf{d}_2^-\mathsf{d}_3^-\,c\Delta^+ + 4\widehat{\partial}_1 d\Delta^+ + \theta\widehat{\Delta}(\mathsf{d}_*^-)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)\Delta^-\Big)\cdot z_+^r[\theta] + \ldots\,.
\end{aligned}
$$

Notice that $\mathsf{d}_2^-\mathsf{d}_3^-\, c = \mathsf{d}_2^-\mathsf{d}_3^-\mathsf{d}_2^+\mathsf{d}_3^-\mathsf{d}_1^- = -\widehat{\partial}_1\mathsf{d}_2^-\mathsf{d}_3^-\mathsf{d}_1^- = -\widehat{\partial}_1\, d$. Therefore

$$e_0\,\mathbf{v}_2 = \Big(3\widehat{\partial}_1 d\Delta^+ + \theta\widehat{\Delta}(\mathsf{d}_*^-)(\mathsf{d}_2^-\mathsf{d}_3^+ + \mathsf{d}_2^+\mathsf{d}_3^-)\Delta^-\Big)\cdot z_+^r[\theta] + \ldots \neq 0$$

We get no singular vectors in this case and we have come an end in the list of cases. Thus we have found all the singular vectors listed and nothing else, this ends the proof of Theorem 8.4. □

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Authors' addresses:

Department of Mathematics, MIT, Cambridge MA 02139, USA,
email: kac@math.mit.edu

Inst. for Matemat. Fag, NTNU, Gløshaugen, 7491 Trondheim, NORWAY,
email: rudakov@math.ntnu.no